On the Normal Form of the Sinh-Gordon Equation

Dissertation

zur

Erlangung der naturwissenschaftlichen Doktorwürde
(Dr. sc. nat.)

vorgelegt der

Mathematisch-naturwissenschaftlichen Fakultät

der

Universität Zürich

von

Yannick Widmer

von

Sumiswald BE

Promotionskommission

Prof. Dr. Thomas Kappeler (Vorsitz)
Prof. Dr. Camillo De Lellis
Prof. Dr. Benjamin Schlein

Zürich, 2018

back of title page

Die vorliegende Arbeit wurde von der Mathematisch-naturwissenschaftlichen Fakultät der Universität Zürich im Frühjahrssemester 2018 als Dissertation angenommen.

Promotionskomitee: Prof. Dr. Thomas Kappeler (Vorsitz), Prof. Dr. Camillo De Lellis,
Prof. Dr. Benjamin Schlein

Contents

1 Introduction

The subject of this work is the sinh-Gordon equation,

$$u_{tt} - u_{xx} = -\sinh u, \quad x \in \mathbb{T} = \mathbb{R}/\mathbb{Z},\ t \in \mathbb{R} \tag{1.1}$$

where u is assumed to be real valued. Note that a purely imaginary solution $u = \mathrm{i}\check{u}$ of (1.1) satisfies $\check{u}_{tt} - \check{u}_{xx} = -\sin\check{u}$, referred to as sine-Gordon equation. Both equations are nonlinear perturbations of the Klein-Gordon equation $u_{tt} - u_{xx} = mu$ (with $m = -1$) and have wide ranging applications in geometry and quantum mechanics (cf. discussion at the end of the introduction). They have been extensively studied. In particular it is known that (1.1) is an integrable PDE. Our aim is to study its normal form and obtain results comparable to the ones for the Korteweg-de Vries equation (cf. [8]) and the defocusing NLS equation (cf. [6]). The normal form of such equations can be used to analyze their solutions and to study their (Hamiltonian) perturbations. First let us recall that the sinh-Gordon equation can be written in Hamiltonian form. To describe it we introduce for any given $s \in \mathbb{R}$ the Sobolev space

$$H^s_{\mathbb{R}} = H^s(\mathbb{T}, \mathbb{R}), \quad H^s_r := H^s_{\mathbb{R}} \times H^s_{\mathbb{R}}.$$

We also write $L^2_{\mathbb{R}}$ for $H^0_{\mathbb{R}}$ and accordingly L^2_r for H^0_r. Equation (1.1) can be written as a system for $u_1 = u$ and $u_2 = u_t$

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_t = X_{H_{\sinh}}, \quad X_{H_{\sinh}}(u_1, u_2) = \begin{pmatrix} u_2 \\ u_{1xx} - \sinh u_1 \end{pmatrix} \tag{1.2}$$

where $X_{H_{\sinh}}$ is the Hamiltonian vector field associated to the Hamiltonian

$$H_{\sinh} = \int_0^1 \frac{1}{2}u_2^2 + \frac{1}{2}(u_{1x})^2 + \cosh u_1 \,\mathrm{d}x, \tag{1.3}$$

with respect to the Poisson bracket

$$\{F, G\}_1 = \int_0^1 \begin{pmatrix} \partial_{u_1} F \\ \partial_{u_2} F \end{pmatrix} J \begin{pmatrix} \partial_{u_1} G \\ \partial_{u_2} G \end{pmatrix} \mathrm{d}x, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{1.4}$$

Here F, G are functionals on some Sobolev space with sufficiently regular $L^2_{\mathbb{R}}$-gradients $(\partial_{u_1} F, \partial_{u_2} F)$, $(\partial_{u_1} G, \partial_{u_2} G)$. Note that H is defined for $(u_1, u_2) \in H^1_{\mathbb{R}} \times L^2_{\mathbb{R}}$. To avoid a phase space consisting of pairs of functions of different regularity we introduce the following coordinates,

$$(q, p) = (u_1, -P^{-1}u_2) \in H^1_r \tag{1.5}$$

where P denotes the Fourier multiplier operator $P := \sqrt{1 - \partial_x^2}$. Note that for any $s \in \mathbb{R}$, P is a linear isomorphism $P : H^s_{\mathbb{R}} \to H^{s-1}_{\mathbb{R}}$ with inverse $P^{-1} : H^{s-1}_{\mathbb{R}} \to H^s_{\mathbb{R}}$. When expressed in these coordinates, equation (1.2) becomes

$$\begin{pmatrix} q_t \\ p_t \end{pmatrix} = -\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} Pq + P^{-1}(\sinh q - q) \\ Pp \end{pmatrix}.$$

The pullback of the Hamiltonian (1.3) again denoted by $H_{\sinh}$, is given by

$$H_{\sinh}(q, p) = \int_0^1 \left(\frac{1}{2}(Pp)^2 + \frac{1}{2}(Pq)^2 + \cosh(q) - \frac{1}{2}q^2 \right) \mathrm{d}x$$

whereas the pullback of the Poisson bracket (1.4) can be computed as

$$\{F, G\} = \int_0^1 \begin{pmatrix} \partial_q F \\ -P^{-1}\partial_p F \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \partial_q G \\ -P^{-1}\partial_p G \end{pmatrix} \mathrm{d}x = -\int_0^1 \begin{pmatrix} \partial_q F \\ \partial_p F \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} P^{-1} \begin{pmatrix} \partial_q G \\ \partial_p G \end{pmatrix}. \tag{1.6}$$

Hence the Hamiltonian vector field of $H_{\sinh}$ is given by

$$X_{H_{\sinh}} = -\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} P^{-1} \begin{pmatrix} P^2 q + \sinh(q) - q \\ P^2 p \end{pmatrix} = \begin{pmatrix} -Pp \\ Pq + P^{-1}(\sinh(q) - q) \end{pmatrix}.$$

Here and in the sequel, we suppress matrix coefficients which vanish, so e.g. $\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$ stays for $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

According to [4], the sinh-Gordon equation admits a Lax pair formulation. More precisely consider for $(q,p) \in H^1_r$ the differential operators

$$Q(q,p) = Q_1\partial_x + Q_0(q,p), \quad K(q,p) = K_1\partial_x + K_0(q,p) \tag{1.7}$$

where the coefficients Q_1, Q_0, K_1, K_0 are the 4×4 matrices given by

$$\begin{aligned} Q_1 &= \begin{pmatrix} -J & \\ & \end{pmatrix}, \quad Q_0(q,p) = \begin{pmatrix} A(q,p) & B(q,p) \\ B(q,p) & \end{pmatrix}, \\ K_1 &= \begin{pmatrix} -I & \\ & I \end{pmatrix}, \quad K_0(q,p) = \begin{pmatrix} & -2JB(q,p) \\ -2B(q,p)J & \end{pmatrix}, \end{aligned}$$

with I, J, R, Z denoting the 2×2 matrices

$$I = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \quad J = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, \quad R = \begin{pmatrix} \mathrm{i} & \\ & -\mathrm{i} \end{pmatrix}, \quad Z = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \tag{1.8}$$

and

$$A(q,p) := -\frac{1}{4}(Pp + q_x)Z, \quad B(q,p) \equiv B(q) := \frac{1}{4}e^{\mathrm{i}Rq/2} = \frac{1}{4}\begin{pmatrix} \exp(-q/2) & \\ & \exp(q/2) \end{pmatrix}. \tag{1.9}$$

One verifies that $t \mapsto (q(t), p(t))$ is a solution of the sinh-Gordon equation (1.1) iff $t \mapsto Q(q(t), p(t))$, $t \mapsto K(q(t), p(t))$ satisfy

$$Q_t = [K, Q]. \tag{1.10}$$

Indeed, using that $[K_1, Q_1] = 0$ and $(K_1)_x = 0$, $(Q_1)_x = 0$ one computes that $[K_1\partial_x, Q_1\partial_x] = 0$ implying that

$$[K, Q] = [K_1\partial_x + K_0, Q_1\partial_x + Q_0] = [K_1, Q_0]\partial_x + K_1(Q_0)_x + [K_0, Q_1]\partial_x - Q_1(K_0)_x + [K_0, Q_0]$$

Since $[K_1, Q_0] + [K_0, Q_1] = 0$ it follows that

$$[K, Q] = K_1(Q_0)_x - Q_1(K_0)_x + [K_0, Q_0]$$

where

$$K_1(Q_0)_x - Q_1(K_0)_x = \begin{pmatrix} -A_x & B_x \\ B_x & \end{pmatrix}$$

and

$$[K_0, Q_0] = \begin{pmatrix} 2B^2J - 2JB^2 & 2AJB \\ -2BJA & \end{pmatrix}.$$

Note that $-BJA = AJB = \frac{1}{16}(Pp + q_x)\begin{pmatrix} e^{-q/2} & \\ & -e^{q/2} \end{pmatrix}$ and $2B^2J - 2JB^2 = -\frac{1}{4}\sinh(q)\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$. Since $Q_t = (Q_0)_t = \begin{pmatrix} A_t & B_t \\ B_t & \end{pmatrix}$ equation (1.10) is equivalent to the following system of equations

$$\begin{aligned} -(Pp + q_x)_t &= (Pp + q_x)_x - \sinh(q) \\ \left(e^{-q/2}\right)_t &= \left(e^{-q/2}\right)_x + \frac{1}{2}(Pp + q_x)\, e^{-q/2}, \end{aligned}$$

which we write as

$$(\partial_t + \partial_x)(Pp + q_x) = \sinh(q) \tag{1.11}$$

$$q_t = -Pp. \tag{1.12}$$

Clearly, $t \mapsto q(t)$, $t \mapsto p(t)$ solve (1.11) - (1.12) iff $t \mapsto q(t)$ satisfies equation (1.1) and $q_t = -Pp$.

The identity (1.10) leads to a family of first integrals of the sinh-Gordon equation. Expressed in a somewhat informal way (i.e., without addressing issues of regularity) it follows from (1.10) that for any solution $t \mapsto (q(t), p(t))$ of the sinh-Gordon equation, the periodic spectrum $\mathrm{spec}_{per} Q(q(t), p(t))$ of the operator $Q(q(t), p(t))$ is independent of t. Hence for any $(q_0, p_0) \in H^1_r$, the set

$$Iso(q_0, p_0) = \{\ (q,p) \in H^1_r \ :\ \mathrm{spec}_{per} Q(q,p) = \mathrm{spec}_{per} Q(q_0, p_0)\ \}$$

is left invariant by the flow of the sinh-Gordon equation. It is referred to as the isospectral set of (q_0, p_0) and its elements as isospectral potentials. As a consequence, any functional F, defined on H_r^1 (or a subset of it), which is constant on any isospectral set, is a first integral of the sinh- Gordon equation. Functionals with this property are referred to as spectral invariants.

In order to state our main results we introduce for any $s \in \mathbb{R}$ the Hilbert space $h_r^s = \ell^{2,s} \times \ell^{2,s}$ and its complexification h_c^s where $\ell^{2,s} = \ell^{2,s}(\mathbb{Z}, \mathbb{R})$ is the weighted ℓ^2 sequence space given by

$$\ell^{2,s} := \{ \, x = (x_n)_n \in \ell^2(\mathbb{Z}, \mathbb{R}) \, : \, \|x\|_s^2 = \sum \langle n \rangle^{2s} |x_n|^2 < \infty \, \},$$

$\ell^{2,s}_{\mathbb{C}} = \ell^{2,s}(\mathbb{Z}, \mathbb{C})$, and $\langle n \rangle = (n^2\pi^2 + 1)^{1/2}$. Correspondingly we denote by $H^s_{\mathbb{C}}$ the Sobolev space $H^s(\mathbb{T}, \mathbb{C})$ and by $H^s_c := H^s_{\mathbb{C}} \times H^s_{\mathbb{C}}$ the complexification of H^s_r. Furthermore we denote by $\ell^{1,s} = \ell^{1,s}(\mathbb{Z}, \mathbb{R})$ the weighted ℓ^1-sequence space

$$\ell^{1,s} := \{ \, x = (x_n)_n \in \ell^1(\mathbb{Z}, \mathbb{R}) \, : \, \|x\|_{1,s} = \sum_{n \in \mathbb{Z}} \langle n \rangle^s |x_n| \, \}$$

and by $\ell^{1,s}_+$ its positive quadrant,

$$\ell^{1,s}_+ := \{ \, x = (x_n)_n \in \ell^{1,s} \, : \, x_n \geq 0 \;\; \forall \, n \in \mathbb{Z} \, \}.$$

Finally we introduce the following weighted version of the Fourier transform $\mathcal{F} : H_c^1 \to h_c^{1/2}$

$$\mathcal{F} : H_c^1 \to h_c^{1/2}, \; (q, p) \mapsto ((\langle 2n \rangle^{1/2} \alpha_n(q,p))_{n \in \mathbb{Z}}, (\langle 2n \rangle^{1/2} \beta_n(q,p)_{n \in \mathbb{Z}})$$

where for any $n \in \mathbb{Z}$

$$\alpha_n(q,p) = \int_0^1 (-q(x) \cos(2n\pi x) + p(x) \sin(2n\pi x)) \, \mathrm{d}x$$

and

$$\beta_n(q,p) = \int_0^1 (-q(x) \sin(2n\pi x) - p(x) \cos(2n\pi x)) \, \mathrm{d}x.$$

Clearly $\mathcal{F} : H_c^1 \to h_c^{1/2}$ and its restriction to H_r^1, again denoted by F, $\mathcal{F} : H_r^1 \to h_r^{1/2}$, are linear isomorphisms.

Our main results are the following ones.

***Theorem* 1.1** *There exists a map Φ, defined on a neighborhood V_0 of 0 in H_r^1 with values in $h_r^{1/2}$ with the following properties.*

(i) $\Phi : V_0 \to \Phi(V_0)$ $v \mapsto ((x_n(v))_{n \in \mathbb{Z}}, \; (y_n(v))_{n \in \mathbb{Z}}))$ *is a real analytic diffeomorphism.* $\Phi(0) = (0,0)$ *and the image $\Phi(V_0)$ is a ball, centered at 0. Its differential $\mathrm{d}_0\Phi$ at 0 is given by the weighted Fourier transform $\mathcal{F}$.*

(ii) *The actions $I_n := (x_n^2 + y_n^2)/2$, $n \in \mathbb{Z}$, are spectral invariants and hence first integrals of the* sinh-*Gordon equation.*

(iii) *The* sinh-*Gordon Hamiltonian when expressed in the coordinates x_n, y_n $n \in \mathbb{Z}$, is in normal form, meaning that it is a function of the actions alone, $H = H((I_n)_{n \in \mathbb{Z}})$. Here H is a real analytic function, defined in a neighborhood of 0 in $\ell^{1,1}_+$.*

(iv) *The* sinh-*Gordon equation on V_0, when expressed in the coordinates x_n, y_n, $n \in \mathbb{Z}$, takes the form*

$$\dot{x}_n = -\omega_n y_n, \quad \dot{y}_n = \omega_n x_n, \qquad \forall \, n \in \mathbb{Z}$$

where $\omega_n := \partial_{I_n} H$, $n \in \mathbb{Z}$, are the sinh-*Gordon frequencies. Hence it can be solved by quadrature,*

$$x_n(t) = \sqrt{2I_n(0)} \cos(-\omega_n t + \theta_n(0)), \quad y_n(t) = \sqrt{2I_n(0)} \sin(-\omega_n t + \theta_n(0))$$

where $I_n(0)$, $\theta_n(0)$ are determined by the initial data, $x_n(0) = \sqrt{2I_n(0)} \cos(\theta_n(0))$, $y_n(0) = \sqrt{2I_n(0)} \sin(\theta_n(0))$ and $((x_n(0))_{n \in \mathbb{Z}}, (y_n(0))_{n \in \mathbb{Z}}) \in \Phi(V_0)$. The solution $((x_n(t))_{n \in \mathbb{Z}}, (y_n(t))_{n \in \mathbb{Z}})$ stays in $\Phi(V_0)$ for all times $t \in \mathbb{R}$.

The coordinates x_n, y_n, $n \in \mathbb{Z}$ are referred to as Birkhoff coordinates and the map Φ as Birkhoff map.

Comments: Let us briefly comment on Theorem 1.1. (i) Actually, the results obtained are more general than the ones stated in Theorem 1.1. The map Φ is defined on a neighborhood W of H^1_r in H^1_c as a real analytic map $\Phi : W \to h^{1/2}_c$ and it is shown that Φ is a local diffeomorphism at any so called finite gap potential. (ii) The representation of solutions of the sinh-Gordon equation in terms of Birkhoff coordinates implies that they are almost periodic in time. (iii) Since Φ is real analytic, it is a diffeomorphism in a complex neighborhood of 0 in H^1_c and can be used to bring also the sine-Gordon equation into normal form near 0. We plan to work this out in a future project. (iv) Many open questions remain. To address them would go beyond the scope of this thesis. In particular, we plan to study in future work whether Φ is a global symplectomorphism or at least a local one everywhere in phase space. Furthermore, we plan to investigate the restriction of Φ to the Sobolev spaces H^s_r for any $s \geq 1$ and to study regularizing properties of $\Phi - \mathcal{F}$.

Related work: The sinh-Gordon and sine-Gordon equations have a long and distinguished history. They first came up in the 19th century in the theory of pseudospherical surfaces (cf [2]). Actually, it is the version of these equations in light cone coordinates, $w_{\xi\eta} = -\sinh w$ and respectively, $w_{\xi\eta} = -\sin w$, that were considered in this context. In view of the many applications, the sinh-Gordon and sine-Gordon equations are very important equations in contemporary physics: they arise in model field theories (see e.g. [20]), superconductivity (see e.g. [16]) and in mechanical models of nonlinear wave propagation (see e.g. [15] and references therein). Various techniques for obtaining solutions of the sinh-Gordon and sine-Gordon equations *on the real line* have been developed, amongst them Lax-pair representation, inverse scattering, the Bäcklund transformation, and the transformation $u = 4\,\mathrm{arctanh}(w)$ or $u = 4\arctan(w)$ (cf e.g. [5], [14] and references therein). Using methods of algebraic geometry, the versions of these equations in light cone coordinates in the class of quasi-periodic functions (with respect to the light cone coordinates) have been extensively studied (cf e.g. [1] and references therein). In our earlier work [21], these equations have been studied in the class of functions which are periodic with respect to one of the two light cone coordinates. The main result in [21] says that in this case, the equations can be viewed as Hamiltonian PDEs with Hamiltonian in the Poisson algebra of the defocusing and respectively focusing mKdV equation, for which a normal form theory has been established. The mentioned results for the version of the sinh-Gordon and sine-Gordon equations in light cone coordinates do not apply to the sinh-Gordon equation (1.1) or the sine-Gordon equation *on the circle* since in light cone coordinates, the periodicity with respect to the x-variable gets lost.

Important contributions to the analysis of equation (1.1) and its normal form were obtained in [17] where parts of the material, discussed in Section 2, 3, and 7 of our work, are presented and the spectral curve associated to the operator $Q(q,p)$, defined in (1.7), is studied. Based on results in [17] and [12], it is shown in [13] that near a class of finite gap potentials, the sine-Gordon equation admits canonical coordinates so that the sine-Gordon Hamiltonian is in normal form up to order three. These coordinates are then used in [13] to prove a KAM type result for a class of small Hamiltonian perturbations of the sine-Gordon equations. In future work we plan to use the Birkhoff coordinates, constructed in this work, to obtain such type of results for the sinh-Gordon equation near *arbitrary* finite gap potentials.

Organisation: In Section 2 we record the results needed on the fundamental solution of the operator $Q(v)$, the main purpose being to introduce notation and to state the results in the form needed later. In Section 3 we study the asymptotics of the periodic, the Dirichlet, and the Neumann eigenvalues of the operator $Q(v)$ for $v \in H^1_c$. Although the material in these two sections is by and large known (cf e.g. [17]), for the convenience of the reader, we included the proofs of the stated results. In Section 4 we prove the summability of the gap lengths in weighted ℓ^2 spaces, introduce the notion of left and right sided finite gap potentials and prove that the sets constituted of such potentials are dense in H^s_c for any $s \geq 1$. In Section 5 we compute the gradient of the fundamental solution as well as of the ones of simple Dirichlet, Neumann, and periodic eigenvalues of the operator $Q(v)$ and establish asymptotics for these gradients. In Section 6 we consider potentials in some small complex neighbourhood of H^1_r in H^1_c, introduce the notion of isolating neighbourhoods, and discuss product representations of various quantities. In Section 7 we compute various Poisson brackets and study isospectral sets. The material covered in Section 7 is by and large known (cf e.g. [17]). It is presented in such a way that it can be readily applied for our purposes (cf Sections 8 and 9). In Section 8 we introduce candidates for action and angle variables and study their properties. In Section 9 we study candidates for Birkhoff coordinates defined in terms of the action-angle variables introduced in Section 8 and prove Theorem 1.1. In particular we show that our candidates provide a system of coordinates near 0 so that the sinh-Gordon Hamiltonian when expressed in these coordinates is in Birkhoff normal form.

2 Fundamental solution

In this chapter we study the fundamental solution of the differential operator $Q = Q(q, p)$ defined in (1.7). The results obtained will be used in particular in Chapter 3 in order to analyze the periodic and the Dirichlet spectrum of Q.

Note that $QF = 0$ if and only if $F = 0$ and for any given $\lambda \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$, a function F in $H^1_{loc}(\mathbb{R}, \mathbb{C}^4)$ is a solution of

$$QF = \lambda F \tag{2.1}$$

if and only if $F = (f, \lambda^{-1}Bf)$ and f satisfies the following first order ODE

$$-J\partial_x f + (A + B^2/\lambda)f = \lambda f. \tag{2.2}$$

Here and in the sequel we often write $\partial_x f$ for f_x and v for $(q, p) \in H^1_c$. Let $M = M(x, \lambda, v) \in \mathbb{C}^{2\times 2}$ be the fundamental solution for equation (2.2), meaning that

$$(-J\partial_x + A(x) + B^2(x)/\lambda)M(x, \lambda, v) = \lambda M(x, \lambda, v), \quad M(0, \lambda, v) = Id, \quad v = (q, p) \in H^1_c.$$

Clearly one has

$$\partial_x M = J\left(\lambda - A - B^2/\lambda\right) M, \tag{2.3}$$

or, taking the definition (1.9) into account

$$\partial_x M(x, \lambda, v) = J\left(\lambda + \frac{1}{4}(Pp(x) + q_x(x))\begin{pmatrix} & 1 \\ 1 & \end{pmatrix} - \frac{1}{16\,\lambda}\begin{pmatrix} e^{-q(x)} & \\ & e^{q(x)} \end{pmatrix}\right) M.$$

Sometimes it is convenient to use the matrix valued function $\mathcal{M}$ instead of M where

$$\mathcal{M}(x, \lambda, v) = TM(x, \lambda, v)T^{-1}, \quad T = \begin{pmatrix} 1 & \mathrm{i} \\ 1 & -\mathrm{i} \end{pmatrix}, \quad T^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -\mathrm{i} & \mathrm{i} \end{pmatrix}. \tag{2.4}$$

Introduce

$$\mathcal{Q} = \begin{pmatrix} T & \\ & T \end{pmatrix} Q \begin{pmatrix} T^{-1} & \\ & T^{-1} \end{pmatrix} = \mathcal{Q}_1\partial_x + \mathcal{Q}_0(v) \tag{2.5}$$

where

$$\mathcal{Q}_1 = \begin{pmatrix} R & \\ & \end{pmatrix}, \quad \mathcal{Q}_0(v) = \begin{pmatrix} \mathcal{A}(v) & \mathcal{B}(v) \\ \mathcal{B}(v) & \end{pmatrix} \tag{2.6}$$

and

$$\mathcal{A}(v) = TA(v)T^{-1} = -\frac{\mathrm{i}}{4}(Pp + q_x)J, \quad \mathcal{B}(v) = TB(v)T^{-1} = \frac{1}{4}\begin{pmatrix} \cosh(q/2) & -\sinh(q/2) \\ -\sinh(q/2) & \cosh(q/2) \end{pmatrix}. \tag{2.7}$$

Note that

$$\mathcal{B}(v)^2 = TB(v)^2T^{-1} = \frac{1}{16}\begin{pmatrix} \cosh(q) & -\sinh(q) \\ -\sinh(q) & \cosh(q) \end{pmatrix} \tag{2.8}$$

F is a solution of $QF = \lambda F$ iff $\mathcal{Q}TF = \lambda TF$. Furthermore $\mathcal{F}$ is a solution of

$$\mathcal{Q}\mathcal{F} = \lambda\mathcal{F} \tag{2.9}$$

iff $\mathcal{F} = (\mathcal{F}_1, \lambda^{-1}\mathcal{B}\mathcal{F}_1)$ and

$$\partial_x\mathcal{F}_1 = -R\left(\lambda - \mathcal{A} - \mathcal{B}^2/\lambda\right)\mathcal{F}_1. \tag{2.10}$$

Hence $\mathcal{M}$ satisfies

$$\partial_x\mathcal{M} = -R\left(\lambda - \mathcal{A} - \mathcal{B}^2/\lambda\right)\mathcal{M}, \quad \mathcal{M}(0, \lambda, v) = I. \tag{2.11}$$

2.1 Symmetries and estimates for the fundamental solution

First we discuss symmetries of the fundamental solution M needed in the sequel.

***Proposition* 2.1** (Symmetries) *For any* $(x, \lambda, v) \in [0, \infty) \times \mathbb{C}^* \times H^1_c$

(i) (Reflection in λ)

$$M(x, -\lambda, v) = -RM(x, \lambda, v)R, \quad \mathcal{M}(x, -\lambda, v) = -Z\mathcal{M}(x, \lambda, v)Z.$$

(ii) (Reciprocity in λ)

$$M\left(x, \frac{1}{16\lambda}, q, p\right) = -Re^{-\mathrm{i}Rq(x)/2}M(x, \lambda, -q, p)e^{\mathrm{i}Rq(0)/2}R.$$

or, solving for $M(x, \lambda, -q, p)$,

$$M(x, \lambda, -q, p) = -Re^{\mathrm{i}Rq(x)/2}M(x, \frac{1}{16\lambda}, q, p)e^{-\mathrm{i}Rq(0)/2}R.$$

(iii) (Conjugation)

$$M(x, \overline{\lambda}, \overline{v}) = \overline{M(x, \lambda, v)}, \quad \mathcal{M}(x, \overline{\lambda}, \overline{v}) = Z\overline{\mathcal{M}(x, \lambda, v)}Z.$$

(iv) (Reflection of v)

$$M(x, \lambda, -v) = JM(x, \lambda, v)J^{-1}, \quad \mathcal{M}(x, \lambda, -v) = R\mathcal{M}(x, \lambda, v)R^{-1}.$$

Proof. (i) First note that $M(x, -\lambda, v)$ and $-RM(x, \lambda, v)R$ coincide at $x = 0$. It suffices to show that they satisfy both the same first order differential equation. By (2.3) $M(x, -\lambda, v)$ satisfies

$$\partial_x M(x, -\lambda, v) = J\Big(-\lambda + \frac{1}{4}(Pp(x) + q_x(x))Z + \frac{1}{16\lambda}e^{\mathrm{i}Rq(x)}\Big)M(x, -\lambda, v)$$

and thus

$$\partial_x\Big(-RM(x, \lambda, v)R\Big) = -RJ\Big(\lambda + \frac{1}{4}(Pp(x) + q_x(x))Z - \frac{1}{16\lambda}e^{\mathrm{i}Rq(x)}\Big)M(x, \lambda, v)R.$$

Since $RJ = -JR$, $RZ = -ZR$ and $Re^{\mathrm{i}Rq} = e^{\mathrm{i}Rq}R$ one concludes that $-RM(x, \lambda, v)R$ satisfies the same equation as $M(x, -\lambda, v)$

$$\partial_x\Big(-RM(x, \lambda, v)R\Big) = J\Big(-\lambda + \frac{1}{4}(Pp(x) + q_x(x))Z + \frac{1}{16\lambda}e^{\mathrm{i}Rq(x)}\Big)\Big(-RM(x, \lambda, v)R\Big)$$

as claimed. Concerning the second identity of item (i) note that

$$Z = -\mathrm{i}TRT^{-1} \tag{2.12}$$

and hence $-Z\mathcal{M}Z = -ZTMT^{-1}Z = TRMRT^{-1}$ which yields the claimed identity for $\mathcal{M}(x, -\lambda, v)$.

(ii) Again note that $M(x, -\frac{1}{16\lambda}, q, p)$ and $e^{-\mathrm{i}Rq(x)/2}M(x, \lambda, -q, p)e^{\mathrm{i}Rq(0)/2}$ coincide at $x = 0$. Hence it suffices to show that they satisfy both the same first order differential equation. By (2.3) $M(x, -\frac{1}{16\lambda}, q, p)$ satisfies

$$\partial_x M(x, -\frac{1}{16\lambda}, q, p) = J\left(-\frac{1}{16\lambda} + \frac{1}{4}(Pp(x) + q_x(x))Z + \lambda e^{\mathrm{i}Rq(x)}\right)M(x, -\frac{1}{16\lambda}, q, p). \tag{2.13}$$

On the other hand

$$\begin{aligned}\partial_x\left(e^{-\mathrm{i}Rq(x)/2}M(x, \lambda, -q, p)\right) =& e^{-\mathrm{i}Rq(x)/2}J\Big(\lambda + \frac{1}{4}(Pp(x) - q_x(x))Z - \frac{1}{16\lambda}e^{-\mathrm{i}Rq(x)}\Big)M(x, \lambda, -q, p)\\ &- \frac{1}{2}q_x(x)\mathrm{i}Re^{-\mathrm{i}Rq(x)/2}M(x, \lambda, -q, p).\end{aligned}$$

Since $e^{-\mathrm{i}Rq/2}J = Je^{\mathrm{i}Rq/2}$, $e^{\mathrm{i}Rq/2}Z = Ze^{-\mathrm{i}Rq/2}$ and $\mathrm{i}R = -JZ$ one gets

$$\partial_x\Big(e^{-\mathrm{i}Rq(x)/2}M(x, \lambda, -q, p)\Big) = J\Big(\lambda e^{\mathrm{i}Rq(x)} + \frac{1}{4}(Pp(x) + q_x(x))Z - \frac{1}{16\lambda}\Big)e^{-\mathrm{i}Rq(x)/2}M(x, \lambda, -q, p). \tag{2.14}$$

Comparing (2.13) and (2.14) one sees that $M\left(x, -\frac{1}{16\lambda}, q, p\right)$ and $e^{-iRq(x)/2}M(x,\lambda,-q,p)e^{\mathrm{i}Rq(0)/2}$ satisfy the same differential equation and hence must coincide. The first identity of item (ii) then follows from (i). The second identity of (ii) is obtained using the first one.

(iii) Take the complex conjugate of (2.3)

$$\partial_x \overline{M(x,\lambda,v)} = J\left(\overline{\lambda} - \overline{A(v)} - \overline{B(v)}^2/\overline{\lambda}\right)\overline{M(x,\lambda,v)}.$$

Since $\overline{A(v)} = A(\overline{v})$, $\overline{B(v)^2} = B(\overline{v})^2$ and $M(0,\lambda,v) = I$ one concludes that

$$\overline{M(x,\lambda,v)} = M(x,\overline{\lambda},\overline{v}).$$

The second identity of item (iii) then also follows since by (2.4) $\overline{T^{-1}} = T^{-1}Z$.

(iv) Note that $M(x,\lambda,-v)$ satisfies

$$\partial_x M(x,\lambda,-v) = J\left(\lambda - \frac{1}{4}(Pp(x)+q_x(x))Z - \frac{1}{16\lambda}e^{-\mathrm{i}Rq(x)}\right)M(x,\lambda,-v) \tag{2.15}$$

while

$$\partial_x JM(x,\lambda,v) = JJ\left(\lambda + \frac{1}{4}(Pp(x)+q_x(x))Z - \frac{1}{16\lambda}e^{\mathrm{i}Rq(x)}\right)M(x,\lambda,v).$$

The first claimed identity then follows from $JZ = -ZJ$ and $Je^{\mathrm{i}Rq} = e^{-\mathrm{i}Rq}J$ and the second one from $TJT^{-1} = -R$. □

To obtain estimates for the fundamental solution of Q we write (2.11) as an integral equation and represent its solution as a series. Here and in the sequel we will use the Euclidean norm for vectors in $\mathbb{C}^2$ and the induced operator norm for matrices.

We write $M = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}$ where $m_j = m_j(x,\lambda,v)$. The norm $|M|$ of M induced by the Hermitian norm on $\mathbb{C}^2$ can be bounded as follows

$$|M| \le 2\max\{|m_1|+|m_2|, |m_3|+|m_4|\}. \tag{2.16}$$

Let us first compute $\mathcal{M}(x,\lambda,0)$. By (2.11) it satisfies $\partial_x\mathcal{M} = -R\left(\lambda - \frac{1}{16\lambda}\right)\mathcal{M}$. For $\lambda \in \mathbb{C}^*$, let

$$\mathcal{E}_\omega(x) := e^{-R\omega x} = \begin{pmatrix} e^{-\mathrm{i}\omega x} & \\ & e^{\mathrm{i}\omega x} \end{pmatrix}, \quad \omega \equiv \omega(\lambda) = \lambda - \frac{1}{16\,\lambda}. \tag{2.17}$$

Then $\partial_x\mathcal{E}_\omega = -R\omega\mathcal{E}_\omega$ and hence $\mathcal{E}_\omega(x) = \mathcal{M}(x,\lambda,0)$. Note that $\omega(\lambda) = 0$ iff $\lambda = \pm\frac{1}{4}$ and

$$\omega(\frac{1}{16\lambda}) = \omega(-\lambda), \quad \omega(-\lambda) = -\omega(\lambda), \quad \forall\, \lambda \in \mathbb{C}^*. \tag{2.18}$$

For $v \in H^1_c$, $\mathcal{M}(x) = \mathcal{M}(x,\lambda,v)$ satisfies the integral equation

$$\begin{aligned}
\mathcal{M}(x) - \mathcal{E}_\omega(x) = \mathcal{E}_\omega(0)\mathcal{M}(x) - \mathcal{E}_\omega(x)\mathcal{M}(0) &= \int_0^x \partial_s\Big(\mathcal{E}_\omega(x-s)\mathcal{M}(s)\Big)\mathrm{d}s \\
&= \int_0^x R\omega\mathcal{E}_\omega(x-s)\mathcal{M}(s) - \mathcal{E}_\omega(x-s)R\Big(\lambda - \mathcal{A} - \mathcal{B}^2/\lambda\Big)\mathcal{M}(s)\mathrm{d}s \\
&= \int_0^x \mathcal{E}_\omega(x-s)R\Big(\omega - \lambda + \mathcal{A} + \mathcal{B}^2/\lambda\Big)\mathcal{M}(s)\mathrm{d}s
\end{aligned}$$

or

$$\mathcal{M}(x) - \mathcal{E}_\omega(x) = \int_0^x \mathcal{E}_\omega(x-s)\Phi_\lambda(s)\mathcal{M}(s)\mathrm{d}s \tag{2.19}$$

where by (2.7)-(2.8)

$$\Phi_\lambda(s) = R(\omega - \lambda + \mathcal{A} + \mathcal{B}^2/\lambda) = R\left(\frac{1}{16\lambda}\begin{pmatrix}\cosh(q)-1 & -\sinh(q) \\ -\sinh(q) & \cosh(q)-1\end{pmatrix} - \frac{\mathrm{i}}{4}(Pp+q_x)J\right).$$

To investigate the regularity of the fundamental solution we use equation (2.19) to find a series representation for $\mathcal{M}$. Let $\mathcal{M}_0(x) = \mathcal{E}_\omega(x)$ and define $\mathcal{M}_{n+1}(x)$ inductively by

$$\mathcal{M}_{n+1}(x) = \int_0^x \mathcal{E}_\omega(x-s)\Phi_\lambda(s)\mathcal{M}_n(s)\mathrm{d}s. \tag{2.20}$$

Using that $\mathcal{E}_\omega(x-s) = \mathcal{E}_\omega(x)\mathcal{E}_\omega(-s)$ one obtains the following identity for $\mathcal{M}_{n+1}$,

$$\begin{aligned}\mathcal{M}_{n+1}(x) &= \int_0^x \mathcal{E}_\omega(x-s)\Phi_\lambda(s)\mathcal{M}_n(s)\mathrm{d}s\\ &= \mathcal{E}_\omega(x)\int_{0\le x_1\le x_2\le\ldots\le x_n\le x} \mathcal{E}_\omega(-x_n)\Phi_\lambda(x_n)\mathcal{E}_\omega(x_n)\cdots\mathcal{E}_\omega(-x_1)\Phi_\lambda(x_1)\mathcal{E}_\omega(x_1)\,\mathrm{d}x_1\cdots\mathrm{d}x_n.\end{aligned}$$

As usual, we denote by $\|q\|_s$ the $H^s_{\mathbb{C}}$-norm of $q = \sum_{k\in\mathbb{Z}}\hat{q}_{2k}e^{2\pi\mathrm{i}kx}$

$$\|q\|_s = \left(\sum_{k\in\mathbb{Z}}|\hat{q}_{2k}|^2\langle 2k\rangle^{2s}\right)^{1/2}, \quad \langle 2k\rangle = \sqrt{1+(2k\pi)^2} \tag{2.21}$$

and write

$$\|v\|_s = \|q\|_s + \|p\|_s, \quad \|v\|_0 \equiv \|v\|_{L^2} = \|q\|_{L^2} + \|p\|_{L^2}.$$

Note that since for any $f\in C^1(\mathbb{T},\mathbb{C})$ there exists $x\in\mathbb{T}$ such that $|f(x)| = \|f\|_{L^2}$ and hence

$$|f(y)| \le |f(x)| + |\int_x^y f'(s)\,\mathrm{d}s| \le \|f\|_{L^2} + \|f'\|_{L^2} \le 2\|f\|_1 \quad \forall\, y\in\mathbb{T},$$

one has

$$\|q\|_{L^\infty} \le 2\|q\|_1. \tag{2.22}$$

Since $C^1(\mathbb{T},\mathbb{C})$ is dense in $H^1_{\mathbb{C}}$ and $H^1_{\mathbb{C}}\subset C(\mathbb{T},\mathbb{C})$, (2.22) holds on $H^1_{\mathbb{C}}$.

For any complex 2×2 matrix $\begin{pmatrix} a & b\\ c & d\end{pmatrix}$ we denote by $|\begin{pmatrix} a & b\\ c & d\end{pmatrix}|$ its operator norm, induced by the standard hermitian norm $|\cdot|$ in $\mathbb{C}^2$. For instance, $|\begin{pmatrix} a & 0\\ 0 & d\end{pmatrix}| = \max(|a|,|d|)$.

***Theorem* 2.2** (Regularity of the fundamental solution) *The series*

$$\mathcal{M}(x) = \sum_{n=0}^{\infty}\mathcal{M}_n(x)$$

with $\mathcal{M}_n(x)$ given by (2.20) converges in $Mat_{2\times 2}(\mathbb{C})$ absolutely, uniformly on bounded closed subsets of $[0,\infty)\times\mathbb{C}^*\times H^1_c$. $\mathcal{M}$ is continuous in x,λ,v and analytic in $v=(q,p)$ and λ as a map with values in the Banach space $C([0,2],\mathbb{C}^{2\times 2})$ of continuous functions with values in $\mathbb{C}^{2\times 2}$, endowed with the supremum norm. It is the unique solution of (2.11),

$$\partial_x\mathcal{M} = -R\left(\lambda - \mathcal{A} - \mathcal{B}^2/\lambda\right)\mathcal{M}, \quad \mathcal{M}(0,\lambda,v) = I,$$

implying that $\mathcal{M}$ and $\partial_x\mathcal{M}$ are analytic in v and λ as maps with values in $L^2([0,2],\mathbb{C}^{2\times 2})$. Furthermore $\mathcal{M}$ satisfies the following estimate for any $0\le x\le 2$, $\lambda\in\mathbb{C}^*$, $v\in H^1_c$,

$$|\mathcal{M}(x,\lambda,v)| \le e^{|\mathrm{Im}\omega|x}\exp\left(e^{2|\mathrm{Im}\omega|x}\left(\frac{x}{|\lambda|}e^{2\|q\|_1} + \sqrt{x}\|v\|_1\right)\right).$$

Proof. Clearly, one has

$$|\mathcal{M}_0(x)| = |\mathcal{E}_\omega(x)| = e^{|\mathrm{Im}\omega|x}.$$

To estimate $\mathcal{M}_{n+1}$ for $n\ge 0$, we first need to estimate $F(x,\lambda,v) := \int_0^x \mathcal{E}_\omega(-s)\Phi_\lambda(s)\mathcal{E}_\omega(s)\,\mathrm{d}s$. Use the bound (2.16) of the matrix norm and the identity

$$\mathcal{E}_\omega(-s)\Phi_\lambda(s)\mathcal{E}_\omega(s) = \frac{\mathrm{i}}{16\lambda}\begin{pmatrix}\cosh(q(s))-1 & -\sinh(q(s))e^{2\mathrm{i}\omega s}\\ \sinh(q(s))e^{-2\mathrm{i}\omega s} & 1-\cosh(q(s))\end{pmatrix} + \frac{1}{4}(Pp(s)+q_x(s))\begin{pmatrix} & e^{2\mathrm{i}\omega s}\\ e^{-2\mathrm{i}\omega s} & \end{pmatrix} \tag{2.23}$$

to conclude that,

$$|F(x,\lambda,v)| \leq \frac{1}{|\lambda|}\max_{\pm}\int_0^x \left(|\cosh(q(s))-1| + |e^{\pm 2i\omega s}\sinh(q(s))|\right)\mathrm{d}s + \max_{\pm}\int_0^x \left|(Pp(s)+q_x(s))e^{\pm 2i\omega s}\right|\mathrm{d}s.$$

Since $|\cosh(q)-1| \leq \sum_{n\geq 1}\frac{1}{(2n)!}|q|^{2n}$ and $|\sinh(q)| \leq \sum_{n\geq 0}\frac{1}{(2n+1)!}|q|^{2n+1}$, one has $|\cosh(q)-1| + |\sinh(q)| \leq e^{|q|}-1$. Using that $|e^{\pm 2i\omega s}| \leq e^{2|\mathrm{Im}\omega|s}$ and $e^{2|\mathrm{Im}\omega|s} \geq 1$ it then follows that

$$|F(x,\lambda,v)| \leq \int_0^x e^{2|\mathrm{Im}\omega|s}\left(e^{|q(s)|}\frac{1}{|\lambda|} + |Pp(s)+q_x(s)|\right)\mathrm{d}s.$$

Combining the estimates (2.22) and

$$\|e^{2|\mathrm{Im}\omega|s}|Pp+q_x|\|_{L^1([0,x])} \leq \sqrt{x}e^{2|\mathrm{Im}\omega|x}\|Pp+q_x\|_{L^2([0,x])} \leq \sqrt{x}e^{2|\mathrm{Im}\omega|x}\|v\|_1, \quad \forall\, x\geq 0 \tag{2.24}$$

one then finally gets that

$$|F(x,\lambda,v)| \leq e^{2|\mathrm{Im}\omega|x}\left(\frac{x}{|\lambda|}e^{2\|q\|_1} + \sqrt{x}\|v\|_1\right). \tag{2.25}$$

Since the matrix norm is sub-multiplicative, one obtains

$$\begin{aligned}|\mathcal{M}_{n+1}(x)| &\leq e^{|\mathrm{Im}\omega|x}\int_{0\leq x_1\leq\ldots\leq x_n\leq x}\prod_{k=1}^n|\mathcal{E}_\omega(-x_k)\Phi_\lambda(x_k)\mathcal{E}_\omega(x_k)|\mathrm{d}x_n\mathrm{d}x_{n-1}\cdots\mathrm{d}x_1\\ &\leq \frac{e^{|\mathrm{Im}\omega|x}}{n!}|F(x,\lambda,v)|^n.\end{aligned}$$

Hence by (2.25), the series converges normally as claimed and one has

$$|\mathcal{M}(x,\lambda,v)| \leq e^{|\mathrm{Im}\omega|x}\exp\left(e^{2|\mathrm{Im}\omega|x}\left(\frac{x}{|\lambda|}e^{2\|q\|_1} + \sqrt{x}\|v\|_1\right)\right), \quad \forall\, x\geq 0.$$

Since for any given $x\geq 0$, $\mathcal{E}_\omega(x)$ and $\Phi_\lambda(x)$ are analytic in $(\lambda,v)\in\mathbb{C}^*\times H^1_c$ and continuous in $(x,\lambda,v)\in[0,\infty)\times\mathbb{C}^*\times H^1_c$ so is $\mathcal{M}_{n+1}$ for any $n\geq 0$ by the definition (2.20) and hence $\mathcal{M}=\sum_{n\geq 0}\mathcal{M}_n$ in view of the normal convergence of the series. It then follows that $\mathcal{M}$ is analytic as a map of v and λ with values in $C([0,2],\mathbb{C}^{2\times 2})$. Finally substituting the series into (2.20) and using that by the normal convergence of the series, sum and integral commute, one gets (2.19). Since $\mathcal{M}(x)$ and $\mathcal{E}_\omega(x)$ are continuous, (2.11) holds in the L^2 sense. It then follows that $\mathcal{M}$ and $\partial_x\mathcal{M}$ are analytic in v and λ as maps with values in $L^2([0,2],Mat_{2\times 2}(\mathbb{C}))$. □

From Theorem 2.2 we derive the following bounds for $M=T^{-1}\mathcal{M}T$.

***Corollary* 2.3** *M is continuous in x,v,λ and for each fixed x, it is analytic in v, and λ. It is the unique solution of (2.3), implying that M and $\partial_x M$ are continuous in v, and λ as maps with values in $L^2([0,2],Mat_{2\times 2}(\mathbb{C}))$. Furthermore M satisfies the following estimates for any $0\leq x\leq 2$, $\lambda\in\mathbb{C}^*$ $v\in H^1_c$,*

$$|M(x,\lambda,v)| \leq e^{|\mathrm{Im}\omega|x}\exp\left(e^{2|\mathrm{Im}\omega|x}\left(\frac{x}{|\lambda|}e^{2\|q\|_1} + \sqrt{x}\|v\|_1\right)\right).$$

and

$$|M(x,\frac{1}{16\lambda},v)| \leq e^{2\|q\|_1+|\mathrm{Im}\omega|x}\exp\left(e^{2|\mathrm{Im}\omega|x}\left(\frac{x}{|\lambda|}e^{2\|q\|_1} + \sqrt{x}\|v\|_1\right)\right).$$

Proof. Since $M=T^{-1}\mathcal{M}T$ the regularity statements follow from Theorem 2.2. Furthermore since $\frac{i}{\sqrt{2}}T$ is unitary one gets

$$|M(x,\lambda,v)| \leq e^{|\mathrm{Im}\omega|x}\exp\left(e^{2|\mathrm{Im}\omega|x}\left(\frac{x}{|\lambda|}e^{2\|q\|_1} + \sqrt{x}\|v\|_1\right)\right), \quad \forall\, x\geq 0.$$

Since $\omega(\frac{1}{16\lambda})=\omega(-\lambda)=-\omega(\lambda)$ and by Proposition 2.1 (ii),

$$M(x,\frac{1}{16\,\lambda},q,p) = -Re^{-iRq(x)/2}M(x,\lambda,-q,p)e^{iRq(0)/2}R,$$

the latter estimate yields

$$\begin{aligned}|M(x,\frac{1}{16\lambda},q,p)| \leq & e^{2\|q\|_1}\left|M(x,\lambda,-q,p)\right| \\ \leq & e^{2\|q\|_1+|\mathrm{Im}\omega|x}\exp\left(e^{2|\mathrm{Im}\omega|x}\left(\frac{x}{|\lambda|}e^{2\|q\|_1}+\sqrt{x}\|v\|_1\right)\right).\end{aligned}$$

□

Next we prove that M is compact in v uniformly on closed bounded sets of $(x,\lambda)\in[0,\infty)\times\mathbb{C}^*$.

***Definition* 2.4** *We call a map from a subset U of a Hilbert space H into some Banach space compact if it maps sequences in U which converge weakly in H, to strongly convergent sequences.*

***Proposition* 2.5** *For any sequence $(v_n)_{n\geq 1}$ in H^1_c which converges weakly to an element v_* in H^1_c as $n\mapsto\infty$, one has $|M(x,\lambda,v_n)-M(x,\lambda,v_*)|\to 0$ as $n\mapsto\infty$, uniformly on closed bounded subsets of $[0,\infty)\times\mathbb{C}^*$.*

Proof. In view of $M=T^{-1}\mathcal{M}T$ it is enough to prove that $\mathcal{M}$ is compact in v uniformly on closed bounded sets of $(x,\lambda)\in[0,\infty)\times\mathbb{C}^*$. In view of the uniform convergence of the series $\sum_{m=0}^{\infty}\mathcal{M}_m(x)$, it suffices to prove the statement for each term $\mathcal{M}_m$. For $M_0=E_\omega$ the statement is obviously true, since this term does not depend on v. Now by induction assume that the statement is true for $\mathcal{M}_m$, and let $(v_n)_{n\geq 0}$ converge weakly to v_* in H^1_c. By equation (2.20) we have

$$\mathcal{M}_{m+1}(x,\lambda,v_n)=\int_0^x\mathcal{E}_\omega(x-s)\Phi(s,\lambda,v_n)\mathcal{M}_m(s,\lambda,v_n)\,\mathrm{d}s. \tag{2.26}$$

By the induction hypothesis

$$|\mathcal{M}_m(x,\lambda,v_n)-\mathcal{M}_m(x,\lambda,v_*)|\to 0$$

uniformly on closed bounded subsets of $[0,\infty)\times\mathbb{C}^*$. Furthermore the weak convergence of v_n in H^1_c implies that $Pp_n+(q_n)_x\rightharpoonup Pp_*+(q_*)_x$ in $L^2([0,x])$ and $q_n\to q_*$, $p_n\to p_*$ in $L^\infty(\mathbb{T})$. It then follows that $\cosh(q_n)\to\cosh(q_*)$, $\sinh(q_n)\to\sinh(q^*)$ in $L^2([0,x])$, yielding that $\mathcal{E}_\omega(x-\cdot)\Phi(\cdot,\lambda,v_n)$ weakly converges to $\mathcal{E}_\omega(x-\cdot)\Phi(\cdot,\lambda,v_*)$ in $L^2([0,x])$, uniformly on bounded subsets of $[0,\infty)\times\mathbb{C}^*$. Hence

$$\int_0^x\mathcal{E}_\omega(x-s)\Phi(s,\lambda,v_n)\mathcal{M}_m(s,\lambda,v_n)\,\mathrm{d}s$$

converges uniformly on closed bounded subsets of $[0,\infty)\times\mathbb{C}^*$ to

$$\int_0^x\mathcal{E}_\omega(x-s)\Phi(s,\lambda,v_*)\mathcal{M}_m(s,\lambda,v_*)\,\mathrm{d}s=\mathcal{M}_{m+1}(x,\lambda,v_*).$$

□

Since $J\left(\lambda-A-B^2/\lambda\right)$ is traceless

$$\det M(x,\lambda,v)=1 \tag{2.27}$$

and hence $M(x,\lambda,v)=\begin{pmatrix}m_1 & m_2\\ m_3 & m_4\end{pmatrix}$ is invertible for any $(x,\lambda,v)\in[0,\infty]\times\mathbb{C}^*\times H^1_c$ and

$$M^{-1}(x,\lambda,v)=\begin{pmatrix}m_4 & -m_2\\ -m_3 & m_1\end{pmatrix},\quad |M^{-1}(x,\lambda,v)|\leq 2\max(|m_4|+|m_2|,|m_3|+|m_1|). \tag{2.28}$$

***Proposition* 2.6** *The λ-derivative $\dot{M}$ of M is given by*

$$\dot{M}(x)=M(x)\int_0^x M^{-1}(s)J(1+B^2/\lambda^2)M(s)\mathrm{d}s.$$

In particular, for any $x\geq 0$, $\dot{M}(x,\lambda,v)$ is analytic on $\mathbb{C}^*\times H^1_c$, and on any closed bounded subset of $[0,\infty)\times\mathbb{C}^*\times H^1_c$ it is compact and bounded.

Proof. Taking the λ-derivative on both sides of equation (2.3) one sees that the λ-derivative $\dot{M}$ of M fulfills

$$\partial_x \dot{M} = J\left(\lambda - A - B^2/\lambda\right)\dot{M} + J(1 + B^2/\lambda^2)M$$

with $\dot{M}(0,\lambda,v)(0) = 0$. The solution of this inhomogeneous linear equation for $\dot{M}$ is then given by

$$\dot{M}(x) = M(x)\int_0^x M^{-1}(s)J(1 + B^2/\lambda^2)M(s)\mathrm{d}s.$$

From this formula and the properties established for M the remaining statements for $\dot{M}$ follow. □

2.2 Asymptotics of the fundamental solution for $|\lambda|$ large and small

In this section we establish bounds for the difference of the fundamental solution M with $M(x,\lambda,0) = E_\omega(x)$ for $|\lambda|$ large and small. First we need to establish the following auxiliary result.

Lemma 2.7 *For any $v \in H_c^1$ and $(x,\lambda) \in [0,1]\times\mathbb{C}^*$, $F(x,\lambda,v) = \int_0^x \mathcal{E}_\omega(-s)\Phi_\lambda(s)\mathcal{E}_\omega(s)\,\mathrm{d}s$ satisfies*

$$|\mathcal{E}_\omega(x)F(x,\lambda,v)| \le \frac{1}{|\lambda|}e^{|\mathrm{Im}\omega(\lambda)|x}e^{2\|q\|_1} + \max_\pm\left|\int_0^x \Big(Pp(s) + q_x(s)\Big)e^{\pm\mathrm{i}\omega(\lambda)(x-2s)}\,\mathrm{d}s\right|.$$

Proof. Multiply (2.23) by $\mathcal{E}_\omega(x)$ to get

$$\begin{aligned}\mathcal{E}_\omega(x)F(x,\lambda,v) = \int_0^x &\frac{\mathrm{i}}{16\lambda}\begin{pmatrix} e^{-\mathrm{i}\omega x}(\cosh(q(s))-1) & -\sinh(q(s))e^{-\mathrm{i}\omega(x-2s)} \\ \sinh(q(s))e^{\mathrm{i}\omega(x-2s)} & e^{\mathrm{i}\omega x}(1-\cosh(q(s)))\end{pmatrix} \\ &+ \frac{1}{4}\Big(Pp(s) + q_x(s)\Big)\begin{pmatrix} & e^{-\mathrm{i}\omega(x-2s)} \\ e^{\mathrm{i}\omega(x-2s)} & \end{pmatrix}\mathrm{d}s.\end{aligned} \tag{2.29}$$

Hence

$$\begin{aligned}\big|\mathcal{E}_\omega(x)F(x,\lambda,v)\big| \le &\int_0^x \frac{1}{|\lambda|}e^{|\mathrm{Im}\omega|x}\Big(|\cosh(q(s))-1| + |\sinh(q(s))|\Big)\,\mathrm{d}x \\ &+ \left|\int_0^x \Big(Pp(s) + q_x(s)\Big)e^{|\mathrm{Im}\omega|(x-2s)}\,\mathrm{d}s\right|.\end{aligned}$$

Using that $|\cosh(q(s))-1| + |\sinh(q(s))| \le e^{2\|q\|_1}$ yields the claim. □

For $(x,\lambda,v) \in [0,1]\times\mathbb{C}^*\times H_c^1$, let

$$\hat{\mathcal{M}}(x,\lambda,v) := \mathcal{M}(x,\lambda,v) - \mathcal{E}_\omega(x).$$

Lemma 2.8 *On $[0,1]\times H_c^1$ for all $\lambda \in \mathbb{C}$ with $|\lambda| \ge 1/4$,*

$$|\hat{\mathcal{M}}(x,\lambda,v)| \le |\mathcal{E}_\omega(x)F(x,\lambda,v)| + C_v e^{|\mathrm{Im}\omega|x}\sqrt{\int_0^x e^{-2|\mathrm{Im}\omega|s}|\mathcal{E}_\omega(s)F(s,\lambda,v)|^2\,\mathrm{d}s}$$

where $C_v = ce^c$ and

$$c = \|v\|_1 + e^{2\|q\|_1}.$$

Proof. By the integral equation (2.19),

$$\begin{aligned}\hat{\mathcal{M}}(x,\lambda,v) &= \int_0^x \mathcal{E}_\omega(x-s)\Phi_\lambda(s)\mathcal{M}(s)\mathrm{d}s \\ &= \int_0^x \mathcal{E}_\omega(x-s)\Phi_\lambda(s)\mathcal{E}_\omega(s)\mathrm{d}s + \int_0^x \mathcal{E}_\omega(x-s)\Phi_\lambda(s)\hat{\mathcal{M}}(s)\mathrm{d}s,\end{aligned}$$

yielding

$$\hat{\mathcal{M}}(x,\lambda,v) = \mathcal{E}_\omega(x)F(x,\lambda,v) + \int_0^x \mathcal{E}_\omega(x-s)\Phi_\lambda(s)\hat{\mathcal{M}}(s)\mathrm{d}s. \tag{2.30}$$

Clearly,

$$\left|\mathcal{E}_\omega(x-s)\Phi_\lambda(s)\hat{\mathcal{M}}(s)\right| \le e^{|\mathrm{Im}\omega|(x-s)}|\Phi_\lambda(s)||\hat{\mathcal{M}}(s)|.$$

It is convenient to introduce the following weighted norm for a x-dependent 2×2 matrix

$$\left|A(x)\right|_\omega := e^{-|\mathrm{Im}\omega|x}|A(x)|.$$

Multiplying both sides of (2.30) by $e^{-|\mathrm{Im}\omega|x}$ one obtains the following estimate

$$\left|\hat{\mathcal{M}}(x)\right|_\omega \le \left|\mathcal{E}_\omega(x)F(x,\lambda,v)\right|_\omega + \int_0^x |\Phi_\lambda(s)|\left|\hat{\mathcal{M}}(s)\right|_\omega \mathrm{d}s$$

and hence by Gronwall's inequality and the estimate $|\Phi(s)| \le \left(|Pp(s)+q_x(s)| + \frac{1}{|\lambda|}e^{|q(s)|}\right) =: b(s,\lambda)$ we get

$$\left|\hat{\mathcal{M}}(x)\right|_\omega \le \left|\mathcal{E}_\omega(x)F(x,\lambda,v)\right|_\omega + \int_0^x \left|\mathcal{E}_\omega(s)F(s,\lambda,v)\right|_\omega b(s,\lambda)e^{\int_s^x b(r,\lambda)\mathrm{d}r}\,\mathrm{d}s.$$

Arguing as in (2.24) with $x=1$, one gets $\int_0^1 b(r,\lambda)\,\mathrm{d}r \le \|v\|_1 + \frac{1}{|\lambda|}e^{2\|q\|_1}$. An application of Cauchy-Schwarz then yields the claim. □

We now use Lemma 2.8 to derive estimates for $\mathcal{M}$ and $\dot{\mathcal{M}}$ from those of F.

***Theorem* 2.9** *$\mathcal{M}$ and $\dot{\mathcal{M}}$ have the following asymptotics:*

(i) *For $|\lambda|\to\infty$, locally uniformly on $[0,1]\times H_c^1$,*

$$\mathcal{M}(x,\lambda,v) = \mathcal{E}_{\omega(\lambda)}(x) + o(e^{|\mathrm{Im}\omega(\lambda)|x}), \quad \dot{\mathcal{M}}(x,\lambda,v) = \dot{\mathcal{E}}_{\omega(\lambda)}(x) + o(e^{|\mathrm{Im}\omega(\lambda)|x}),$$

where $\dot{\mathcal{E}}_{\omega(\lambda)}(x) = -\mathrm{i}\dot{\omega}(\lambda)R\mathcal{E}_{\omega(\lambda)}(x)$ and $\dot{\omega}(\lambda) = 1 + \frac{1}{16\lambda^2}$.

(ii) *For $|\lambda|\to\infty$, uniformly on $[0,1]$ and on bounded subsets of H_c^2,*

$$\mathcal{M}(x,\lambda,v) = \mathcal{E}_{\omega(\lambda)}(x) + O(e^{|\mathrm{Im}\omega(\lambda)|x}/|\omega(\lambda)|), \quad \dot{\mathcal{M}}(x,\lambda,v) = \dot{\mathcal{E}}_{\omega(\lambda)}(x) + O(e^{|\mathrm{Im}\omega(\lambda)|x}/|\omega(\lambda)|).$$

Proof. (i) In view of Lemma 2.8 it remains to prove an appropriate asymptotic estimate for $|\mathcal{E}_\lambda(x)F(x,\lambda,v)|$. By Lemma 2.7

$$|\mathcal{E}_{\omega(\lambda)}(x)F(x,\lambda,v)| \le \frac{1}{|\lambda|}e^{|\mathrm{Im}\omega(\lambda)|x}e^{2\|q\|_1} + \max_{\pm}\left|\int_0^x \Big(Pp(s)+q_x(s)\Big)e^{\pm\mathrm{i}\omega(\lambda)(x-2s)}\,\mathrm{d}s\right|.$$

Apply Lemma E.1 to see that for arbitrary $\epsilon>0$ there is $\omega_\epsilon>0$, depending locally uniformly on $v\in H_c^1$, such that for any $\lambda\in\mathbb{C}^*$ with $|\omega(\lambda)|>\omega_\epsilon$ and $0\le x\le 1$ one has

$$\left|\int_0^x \Big(Pp(s)+q_x(s)\Big)e^{\pm\mathrm{i}\omega(x-2s)}\,\mathrm{d}s\right| \le \epsilon e^{|\mathrm{Im}\omega|x},$$

yielding the stated asymptotics of $\mathcal{M}$. The claimed asymptotics for $\dot{\mathcal{M}}$ is obtained by applying Cauchy's estimate to the λ-derivative of $\hat{\mathcal{M}}$.

(ii) In case $v\in H_c^2$

$$\int_0^x \Big(Pp(s)+q_x(s)\Big)e^{\pm\mathrm{i}\omega(x-2s)}\,\mathrm{d}s$$

can be integrated by parts. Using that for $\omega\equiv\omega(\lambda)\ne 0$, $e^{\pm\mathrm{i}\omega(x-2s)} = \frac{-1}{\pm2\mathrm{i}\omega}\frac{\mathrm{d}}{\mathrm{d}s}e^{\pm\mathrm{i}\omega(x-2s)}$ one gets

$$-\frac{1}{\pm2\mathrm{i}\omega}\left(\Big(Pp(x)+q_x(x)\Big)e^{\mp\mathrm{i}\omega x} + \Big(Pp(0)+q_x(0)\Big)e^{\pm\mathrm{i}\omega x} - \int_0^x \Big(Pp_x(s)+q_{xx}(s)\Big)e^{\pm\mathrm{i}\omega(x-2s)}\,\mathrm{d}s\right).$$

Since by (2.22), $\|q_x\|_{L^\infty}\le 2\|q\|_2$ and $\|Pp\|_{L^\infty}\le 2\|Pp\|_1 = 2\left(\sum_k\langle 2k\rangle^2|\langle 2k\rangle\hat{p}_{2k}|^2\right)^{1/2} = 2\|p\|_2$ one gets for any $0\le x\le 1$, $|\omega(\lambda)|>0$ and $v\in H_c^2$

$$|\mathcal{E}_{\omega(\lambda)}(x)F(x,\lambda,v)| \le \frac{1}{|\lambda|}e^{|\mathrm{Im}\omega(\lambda)|x}e^{2\|q\|_1} + \frac{1}{2|\omega(\lambda)|}\left(2\|v\|_2e^{|\mathrm{Im}\omega(\lambda)|x} + 2\|v\|_2e^{|\mathrm{Im}\omega(\lambda)|x}\right).$$

The claimed asymptotics for $\dot{\mathcal{M}}$ is once more obtained by applying Cauchy's estimate to the λ-derivative of $\hat{\mathcal{M}}$. □

***Theorem* 2.10** *For bi-infinite sequences of complex numbers $(\zeta_n)_n \subset \mathbb{C}^*$ with $|\zeta_n| \geq \frac{1}{4}$, the following holds:*

(i) *If $\zeta_n = n\pi + O(1)$*

$$\mathcal{M}(x, \zeta_n, v) = \mathcal{E}_{\omega(\zeta_n)}(x) + \ell^2_n, \qquad \dot{\mathcal{M}}(x, \zeta_n, v) = \dot{\mathcal{E}}_{\omega(\zeta_n)}(x) + \ell^2_n$$

where $\mathcal{E}_{\omega(\zeta_n)}(x) = e^{-R\omega(\zeta_n)x}$ and $\dot{\omega}(\zeta_n) = 1 + \frac{1}{16\zeta_n^2}$, implying that

$$\dot{\mathcal{E}}_{\omega(\zeta_n)}(x) = -x\dot{\omega}(\zeta_n)Re^{-R\omega(\zeta_n)x} = -xR\mathcal{E}_{\omega(\zeta_n)}(x) + \ell^1_n.$$

These estimates hold uniformly on $[0, 1]$, on bounded subsets of H^1_c and on subsets of sequences $(\zeta_n)_n$ where $(\omega(\zeta_n) - n\pi)_n$ is bounded in $\ell^\infty_{\mathbb{C}}$. In more detail, e.g. the first estimate means that for any bounded subset $V \subset H^1_c$ and any subset B of sequences $(\zeta_n)_n \subset \mathbb{C}^*$, with $(\omega(\zeta_n) - n\pi)_n$ bounded in $\ell^\infty_{\mathbb{C}}$ there exists $C > 0$ so that

$$\sup_{0\leq x\leq 1} \sum_{n\in\mathbb{Z}} \left|\mathcal{M}(x, \zeta_n, v) - \mathcal{E}_{\omega(\zeta_n)}(x)\right|^2 \leq C,$$

for any $v \in V$ and $(\zeta_n)_n \in B$.

(ii) *If $\zeta_n = n\pi + \ell^2_n$, then*

$$\mathcal{M}(x, \zeta_n, v) = \mathcal{E}_{n\pi}(x) + \ell^2_n, \qquad \dot{\mathcal{M}}(x, \zeta_n, v) = -xR\mathcal{E}_{n\pi}(x) + \ell^2_n.$$

These estimates hold uniformly on $[0, 1]$, on bounded subsets of H^1_c, and on subsets of sequences $(\zeta_n)_n$ where $(\omega(\zeta_n) - n\pi)_n$ is bounded in $\ell^2_{\mathbb{C}}$.

Proof. (i) By Lemma 2.8, on $[0, 1] \times H^1_c$ for any $|\lambda| \geq 1/4$

$$|\mathcal{M}(x, \lambda, v) - \mathcal{E}_\omega(x)| \leq O(e^{|\mathrm{Im}\omega|}\|\mathcal{E}_\omega(\cdot)F(\cdot, \lambda, v)\|_{L^\infty([0,1])}).$$

By Lemma 2.7

$$|\mathcal{E}_{\omega(\zeta_n)}(x)F(x, \zeta_n, v)| \leq \frac{1}{|\zeta_n|}e^{|\mathrm{Im}\omega(\zeta_n)|x}e^{2\|q\|_1} + \max_\pm \left|\int_0^x \Big(Pp(s) + q_x(s)\Big)e^{\pm \mathrm{i}\omega(\zeta_n)(x-2s)}\,\mathrm{d}s\right|$$

and by Lemma E.2

$$\sum_{n\in\mathbb{Z}} \max_\pm \left|\int_0^x \Big(Pp(s) + q_x(s)\Big)e^{\pm \mathrm{i}\omega(\zeta_n)(x-2s)}\,\mathrm{d}s\right|^2 \leq 2e^{2b}\|v\|_1, \qquad \forall\ 0 \leq x \leq 1,$$

where $b = \sup_{n\in\mathbb{Z}} |\omega(\zeta_n) - n\pi|$. Altogether, we thus have proved

$$\sup_{0\leq x\leq 1} \sum_{n\in\mathbb{Z}} |\mathcal{M}(x, \zeta_n, v) - \mathcal{E}_{\omega(\zeta_n)}(x)|^2 < \infty.$$

In view of Lemma 2.8, the latter estimate holds uniformly on bounded subsets of H^1_c and on subsets of sequences $(\zeta_n)_n$ in $\mathbb{C}^*$ so that $(\omega(\zeta_n) - n\pi)_n$ is bounded in $\ell^\infty_{\mathbb{C}}$. To obtain the claimed estimate for $\dot{\mathcal{M}}(x, \zeta_n, v) - \dot{\mathcal{E}}_{\omega(\zeta_n)}(x)$ we apply Cauchy's estimate to λ-discs D_{ζ_n} of fixed radius around each ζ_n to get

$$\left|\dot{\mathcal{M}}(x, \zeta_n, v) - \dot{\mathcal{E}}_{\omega(\zeta_n)}(x)\right| \leq O(\sup_{\lambda\in D_{\zeta_n}} |\mathcal{M}(x, \lambda, v) - \mathcal{E}_{\omega(\lambda)}(x)|).$$

Since the radi of the discs D_{ζ_n} do not depend on n one then concludes that

$$\sup_{0\leq x\leq 1} \sum_{n\in\mathbb{Z}} |\dot{\mathcal{M}}(x, \zeta_n, v) - \dot{\mathcal{E}}_{\omega(\zeta_n)}(x)|^2 < \infty$$

where the estimate holds uniformly in the claimed sense. Altogether this establishes the first claim.
(ii) For sequences $(\zeta_n)_n$ with the stronger asymptotics $\omega(\zeta_n) = n\pi + \ell^2_n$, we have

$$\mathcal{E}_{\omega(\zeta_n)}(x) = e^{-R\omega(\zeta_n)x} = e^{-Rn\pi x} + \ell^2_n = \mathcal{E}_{\omega(n\pi)}(x) + \ell^2_n$$

implying the claimed estimate. □

***Theorem* 2.11** *Uniformly on bounded subsets of H^2_c and on subsets of bi-infinite sequences of complex numbers with $\zeta_n = n\pi + O(1)$, $|\zeta_n| \geq \frac{1}{4}$*

$$\sup_{0\leq x\leq 1} \left|\mathcal{M}(x,\zeta_n,v) - \mathcal{E}_{\omega(\zeta_n)}(x)\right| = O\left(\frac{1}{n}\right), \quad \sup_{0\leq x\leq 1} \left|\dot{\mathcal{M}}(x,\zeta_n,v) - \dot{\mathcal{E}}_{\omega(\zeta_n)}(x)\right| = O\left(\frac{1}{n}\right).$$

If in addition $\zeta_n = n\pi + \ell^2_n$, then

$$\sup_{0\leq x\leq 1} \left|\mathcal{M}(x,\zeta_n,v) - \mathcal{E}_{n\pi}(x)\right| = \ell^2_n, \quad \sup_{0\leq x\leq 1} \left|\dot{\mathcal{M}}(x,\zeta_n,v) - \dot{\mathcal{E}}_{n\pi}(x)\right| = \ell^2_n.$$

Note that $\dot{\mathcal{E}}_{\omega(\zeta_n)} = -x\dot{\omega}(\zeta_n)Re^{-R\omega(\zeta_n)x} = -xR\mathcal{E}_{\omega(\zeta_n)}(x) + O\left(\frac{1}{n^2}\right)$.

Proof. The first estimate is a consequence of Theorem 2.9 and the second one is then obtained by applying Cauchy's estimate. If the sequence ζ_n satisfies in addition $\zeta_n = n\pi + \ell^2_n$ then

$$\mathcal{E}_{\omega(\zeta_n)}(x) = \mathcal{E}_{\omega(n\pi)}(x) + \ell^2_n = \mathcal{E}_{n\pi}(x) + \ell^2_n.$$

□

For later reference we state the asymptotics of $M = T^{-1}\mathcal{M}T$, corresponding to the ones obtained for $\mathcal{M}$. Introduce

$$E_{\omega(\lambda)}(x) := M(x,\lambda,0) = T^{-1}\mathcal{E}_{\omega(\lambda)}(x)T = \begin{pmatrix} \cos(\omega(\lambda)x) & \sin(\omega(\lambda)x) \\ -\sin(\omega(\lambda)x) & \cos(\omega(\lambda)x) \end{pmatrix}. \tag{2.31}$$

Since

$$\dot{E}_{\omega(\lambda)}(x) = -T^{-1}x\dot{\omega}(\lambda)Re^{-R\omega(\lambda)x}T = -x\dot{\omega}(\lambda)T^{-1}RTE_{\omega(\lambda)}(x) = xJE_{\omega(\lambda)}(x) + O(e^{|\mathrm{Im}\,\omega(\lambda)|x}/|\lambda|^2),$$

Theorem 2.9 - Theorem 2.11 then yield the following results.

***Theorem* 2.12** *M and $\dot{M}$ have the following asymptotics:*

(i) *For $|\lambda| \to \infty$ locally uniformly on $[0,1] \times H^1_c$,*

$$M(x,\lambda,v) = E_{\omega(\lambda)}(x) + o(e^{|\mathrm{Im}\,\omega(\lambda)|x}), \quad \dot{M}(x,\lambda,v) = xJE_{\omega(\lambda)}(x) + o(e^{|\mathrm{Im}\,\omega(\lambda)|x}).$$

(ii) *For $|\lambda| \to \infty$ uniformly on bounded subsets of $[0,1] \times H^2_c$,*

$$M(x,\lambda,v) = E_{\omega(\lambda)}(x) + O(e^{|\mathrm{Im}\,\omega(\lambda)|x}/|\omega(\lambda)|), \quad \dot{M}(x,\lambda,v) = xJE_{\omega(\lambda)}(x) + O(e^{|\mathrm{Im}\,\omega(\lambda)|x}/|\omega(\lambda)|).$$

Furthermore for bi-infinite sequences of complex numbers $(\zeta_n)_n \subset \mathbb{C}^*$ with $|\zeta_n| \geq \frac{1}{4}$ one has:

(iii) *If $\zeta_n = n\pi + O(1)$, then*

$$M(x,\zeta_n,v) = E_{\omega(\zeta_n)}(x) + \ell^2_n, \qquad \dot{M}(x,\zeta_n,v) = xJE_{\omega(\zeta_n)}(x) + \ell^2_n,$$

uniformly on $[0,1]$, on bounded subsets of H^1_c, and on subsets of sequences $(\zeta_n)_n$ where $(\omega(\zeta_n)-n\pi)_n$ is bounded in $\ell^\infty_{\mathbb{C}}$.

(iv) *If $\zeta_n = n\pi + \ell^2_n$, then*

$$M(x,\zeta_n,v) = E_{n\pi}(x) + \ell^2_n, \qquad \dot{M}(x,\zeta_n,v) = xJE_{n\pi}(x) + \ell^2_n,$$

uniformly on $[0,1]$, on bounded subsets of H^1_c, and on subsets of sequences $(\zeta_n)_n$ where $(\omega(\zeta_n)-n\pi)_n$ is bounded in $\ell^2_{\mathbb{C}}$.

(v) *If $\zeta_n = n\pi + O(1)$ and $v \in H^2_c$, then*

$$M(x,\zeta_n,v) = E_{\omega(\zeta_n)}(x) + O\left(\frac{1}{n}\right), \quad \dot{M}(x,\zeta_n,v) = xJE_{\omega(\zeta_n)}(x) + O\left(\frac{1}{n}\right),$$

uniformly on $[0,1]$, on bounded subsets of H^2_c, and on subsets of sequences ζ_n where $(\zeta_n - n\pi)_n$ is bounded in $\ell^\infty_{\mathbb{C}}$.

Recall that by (2.28), $M^{-1} = \begin{pmatrix} m_4 & -m_2 \\ -m_3 & m_1 \end{pmatrix}$. Furthermore, $E_{\omega(\lambda)}^{-1} = E_{-\omega(\lambda)} = E_{\omega(-\lambda)}$ and hence $(E_{\omega(\lambda)}^{-1})^{\boldsymbol{\cdot}} = -\dot{E}_{\omega(-\lambda)}(x) = -xJE_{\omega(-\lambda)}(x) + O(e^{|\mathrm{Im}\,\omega(-\lambda)|x}/\lambda^2)$. Theorem 2.12 then leads to the following results for M^{-1}.

***Corollary* 2.13** *M^{-1} and $(M^{-1})^{\boldsymbol{\cdot}}$ satisfy the following estimates:*

(i) *For $|\lambda| \to \infty$ locally uniformly on $[0,1] \times H_c^1$,*

$$M^{-1}(x,\lambda,v) = E_{\omega(-\lambda)}(x) + o(e^{|\mathrm{Im}\,\omega(\lambda)|x}) \quad \textit{and} \quad (M^{-1})^{\boldsymbol{\cdot}}(x,\lambda,v) = -xJE_{\omega(-\lambda)}(x) + o(e^{|\mathrm{Im}\,\omega(\lambda)|x})$$

(ii) *For $|\lambda| \to \infty$ uniformly on bounded subsets of $[0,1] \times H_c^2$,*

$$M^{-1}(x,\lambda,v) = E_{\omega(-\lambda)}(x) + o(e^{|\mathrm{Im}\,\omega(\lambda)|x}/|\omega(\lambda)|)$$

and

$$(M^{-1})^{\boldsymbol{\cdot}}(x,\lambda,v) = -xJE_{\omega(-\lambda)}(x) + o(e^{|\mathrm{Im}\,\omega(\lambda)|x}/|\omega(\lambda)|).$$

Furthermore for bi-infinite sequences of complex numbers $(\zeta_n)_n$ in $\mathbb{C}^*$ with $|\zeta_n| \geq \frac{1}{4}$ one has:

(iii) *If $\zeta_n = n\pi + O(1)$, then*

$$M^{-1}(x,\zeta_n,v) = E_{\omega(-\zeta_n)}(x) + \ell_n^2, \qquad (M^{-1})^{\boldsymbol{\cdot}}(x,\zeta_n,v) = -xJE_{\omega(-\zeta_n)}(x) + \ell_n^2,$$

uniformly on $[0,1]$, on bounded subsets of H_c^1, and on subsets of sequences $(\zeta_n)_n$ where $(\omega(\zeta_n)-n\pi)_n$ is bounded in $\ell_{\mathbb{C}}^\infty$.

(iv) *If $\zeta_n = n\pi + \ell_n^2$, then*

$$M^{-1}(x,\zeta_n,v) = E_{-n\pi}(x) + \ell_n^2, \qquad (M^{-1})^{\boldsymbol{\cdot}}(x,\zeta_n,v) = -xJE_{-n\pi}(x) + \ell_n^2,$$

uniformly on $[0,1]$, on bounded subsets of H_c^1, and on subsets of sequences $(\zeta_n)_n$ where $(\omega(\zeta_n)-n\pi)_n$ is bounded in $\ell_{\mathbb{C}}^2$.

(v) *If $\zeta_n = n\pi + O(1)$ and $v \in H_c^2$, then*

$$M^{-1}(x,\zeta_n,v) = E_{\omega(-\zeta_n)}(x) + O\left(\frac{1}{n}\right), \quad (M^{-1})^{\boldsymbol{\cdot}}(x,\zeta_n,v) = -xJE_{\omega(-\zeta_n)}(x) + O\left(\frac{1}{n}\right),$$

uniformly on $[0,1]$, on bounded subsets of H_c^2, and sequences $(\zeta_n)_n$ where $(\zeta_n - n\pi)_n$ is bounded in $\ell_{\mathbb{C}}^\infty$.

2.3 Discriminant and anti-discriminant

In order to study the periodic spectrum of the operator Q, its discriminant plays an important role. For any $v \in H_c^1$, let

$$\grave{M}(\lambda,v) := M(x,\lambda,v)|_{x=1} \quad \text{and} \quad \grave{M}(\lambda,v) =: \begin{pmatrix} \grave{m}_1 & \grave{m}_2 \\ \grave{m}_3 & \grave{m}_4 \end{pmatrix} \tag{2.32}$$

as well as $\grave{\mathcal{M}}(\lambda,v) := \mathcal{M}(1,\lambda,v)$. The discriminant and anti-discriminant are then defined as follows

$$\Delta(\lambda,v) := \frac{1}{2}\mathrm{tr}\grave{M}(\lambda,v) = \frac{1}{2}\mathrm{tr}\grave{\mathcal{M}}(\lambda,v), \quad \delta(\lambda,v) := (\grave{m}_1(\lambda,v) - \grave{m}_4(\lambda,v))/2. \tag{2.33}$$

***Lemma* 2.14** *Δ and δ are analytic maps on $\mathbb{C}^* \times H_c^1$ and have the following symmetries: for any $\lambda \in \mathbb{C}^*$ and $v \in H_c^1$*

(i) *(Reflection in λ) $\Delta(-\lambda,v) = \Delta(\lambda,v)$, $\delta(-\lambda,v) = \delta(\lambda,v)$.*

(ii) *(Reciprocity in λ) $\Delta(\frac{1}{16\lambda},q,p) = \Delta(\lambda,-q,p)$, $\delta(\frac{1}{16\lambda},q,p) = \delta(\lambda,-q,p)$.*

(iii) *(Conjugation) $\Delta(\overline{\lambda},\overline{v}) = \overline{\Delta(\lambda,v)}$, $\delta(\overline{\lambda},\overline{v}) = \overline{\delta(\lambda,v)}$.*

(iv) *(Reflection of v) $\Delta(\lambda,-v) = \Delta(\lambda,v)$, $\delta(\lambda,-v) = -\delta(\lambda,v)$.*

(v) *(Real potentials) If the components q and p of v are real valued then $\Delta(\lambda, v)$ and $\delta(\lambda, v)$ are real for any $\lambda \in \mathbb{R} \cup \mathrm{i}\mathbb{R}$.*

(vi) *(Purely imaginary potentials) If q and p take values in $\mathrm{i}\mathbb{R}$ then $\Delta(\lambda, v)$ is real and $\delta(\lambda, v)$ is purely imaginary for any $\lambda \in \mathbb{R} \cup \mathrm{i}\mathbb{R}$.*

Proof. Items (i) - (iv) follow from Proposition 2.1.
(v) By item (i) and (iii) one has for v real that for any $\lambda \in \mathbb{R} \cup \mathrm{i}\mathbb{R}$

$$\overline{\Delta(\lambda, v)} = \overline{\Delta(\overline{\lambda}, \overline{v})} = \Delta(\lambda, v) \quad \text{and} \quad \overline{\delta(\lambda, v)} = \overline{\delta(\overline{\lambda}, \overline{v})} = \delta(\lambda, v)$$

(vi) In case v is purely imaginary it follows from (i) and (iii)-(iv) that for any $\lambda \in \mathbb{R} \cup \mathrm{i}\mathbb{R}$

$$\overline{\Delta(\lambda, v)} = \overline{\Delta(\overline{\lambda}, -\overline{v})} = \Delta(\lambda, v) \quad \text{and} \quad \overline{\delta(\lambda, v)} = \overline{\delta(\overline{\lambda}, -\overline{v})} = -\delta(\lambda, v).$$

□

The latter lemma and the results of Section 2.1 yield the following.

***Corollary* 2.15** *Discriminant and anti-discriminant together with their λ-derivatives are real analytic on $\mathbb{C}^* \times H^1_c$. On any closed, bounded subset of $\mathbb{C}^* \times H^1_c$, Δ and δ are compact and bounded. More precisely, for any compact subset $K \subset \mathbb{C}^*$ and any closed, bounded subset $V \subset H^1_c$, the map $V \to L^\infty_{\mathbb{C}}(K), v \mapsto (\lambda \mapsto \Delta(\lambda, v))$ is compact in the sense of Definition 2.4.*

For later reference we record the following formulas for $\Delta(\lambda, v)$ and $\delta(\lambda, v)$ at the zero potential $v = 0$. Recall that $\omega(\lambda) = \lambda - \frac{1}{16\,\lambda}$. Taking into account that by (2.31) $\grave{M}(\lambda, 0) = E_{\omega(\lambda)}(1)$, the following holds.

***Lemma* 2.16** *For any $\lambda \in \mathbb{C}^*$*

$$\Delta(\lambda, 0) = \cos(\omega(\lambda)), \quad \dot{\Delta}(\lambda, 0) = -\left(1 + \frac{1}{16\lambda^2}\right)\sin(\omega(\lambda)), \quad \delta(\lambda, 0) = \dot{\delta}(\lambda, 0) = 0.$$

As a consequence $\Delta^2(\lambda, 0) - 1 = -\sin^2(\omega(\lambda))$.

To obtain first rough asymptotics of the periodic eigenvalues we need to compare $\Delta(\lambda, v)$ with $\Delta(\lambda, 0)$. Recall that the domains D_n, $n \geq 0$ where introduced in (3.1).

***Lemma* 2.17** *For any given $v \in H^1_c$, the following asymptotics on $\mathbb{C} \setminus \bigcup_{n\geq 1} D_n \cup (-D_n)$ hold for $|\lambda| \to \infty$*

$$\Delta^2(\lambda) - 1 = -\sin^2(\omega(\lambda))\,(1 + o(1)) = -\sin^2(\lambda)\,(1 + o(1))\,, \tag{2.34}$$

$$\dot{\Delta}(\lambda) = -\sin(\lambda)\,(1 + o(1))\,. \tag{2.35}$$

These estimates hold locally uniformly on H^1_c.

Proof. By Theorem 2.12(i), $\Delta(\lambda, v) = \cos(\omega(\lambda)) + o(e^{|\mathrm{Im}\omega(\lambda)|})$, and thus

$$\Delta^2(\lambda, v) - 1 = -\sin^2(\omega(\lambda))\left(1 + \frac{o(e^{|\mathrm{Im}\omega(\lambda)|})\cos(\omega(\lambda))}{\sin^2(\omega(\lambda))} + \frac{o(e^{2|\mathrm{Im}\omega(\lambda)|})}{\sin^2(\omega(\lambda))}\right). \tag{2.36}$$

For $\lambda \in \mathbb{C}^* \setminus \bigcup_{n\geq 1} D_n \cup (-D_n)$ there exists $m \in \mathbb{Z}$, with $m\pi + \pi/3 \leq \mathrm{Re}\lambda \leq (m+1)\pi - \pi/3$. If in addition, λ is sufficiently large, then $\omega(\lambda) = \lambda - \frac{1}{16\lambda}$ satisfies $m\pi + \pi/4 \leq \mathrm{Re}\omega(\lambda) \leq (m+1)\pi - \pi/4$. Hence $|\sin(\mathrm{Re}\omega(\lambda))| \geq \frac{1}{\sqrt{2}}$ and

$$\begin{aligned}
|\sin(\omega(\lambda))| &= |\sin(\mathrm{Re}\omega(\lambda))\cos(\mathrm{iIm}\omega(\lambda)) + \cos(\mathrm{Re}\omega(\lambda)\sin(\mathrm{iIm}\omega(\lambda))| \\
&= |\sin(\mathrm{Re}\omega(\lambda))\cosh(\mathrm{Im}\omega(\lambda)) + \mathrm{i}\cos(\mathrm{Re}\omega(\lambda)\sinh(\mathrm{Im}\omega(\lambda))| \\
&\geq \frac{1}{\sqrt{2}}\cosh(|\mathrm{Im}\omega(\lambda)|) \geq \frac{1}{\sqrt{2}}e^{|\mathrm{Im}\omega(\lambda)|}.
\end{aligned}$$

It follows that for $\lambda \in \mathbb{C}^* \setminus \bigcup_{n\geq 1} D_n \cup (-D_n)$ sufficiently large

$$\left|\frac{\cos(\omega(\lambda))}{\sin(\omega(\lambda))}\right| \leq \frac{e^{|\mathrm{Im}\omega(\lambda)|}}{|\sin(\omega(\lambda))|} \leq \sqrt{2},$$

and hence the expression inside the large parentheses of (2.36) is $1 + o(1)$. The asymptotics (2.34) then follow since

$$\sin(\omega(\lambda)) = \sin(\lambda)\cos(\frac{1}{16\lambda}) - \sin(\frac{1}{16\lambda})\cos(\lambda) = \sin(\lambda)\left(1 + O(\frac{1}{\lambda^2}) + O(\frac{1}{\lambda})\frac{\cos(\lambda)}{\sin(\lambda)}\right)$$

and

$$\left|\frac{\cos(\lambda)}{\sin(\lambda)}\right| \le \frac{e^{|\mathrm{Im}\lambda|}}{\frac{1}{\sqrt{2}}e^{|\mathrm{Im}\lambda|}} \le \sqrt{2}.$$

Concerning (2.35) note that by Theorem 2.12(i),

$$\dot{M}(x,\lambda,v) = xJE_{\omega(\lambda)}(x) + o(e^{|\mathrm{Im}\omega(\lambda)|x})$$

implying that

$$\dot{\Delta}(\lambda,v) = -\sin(\omega(\lambda)) + o(e^{|\mathrm{Im}\omega(\lambda)|}).$$

A similar argument as the one above then yields the claimed asymptotics. □

2.4 Asymptotics of the discriminant

In this section we describe the asymptotics of $\Delta(\lambda)$ as $|\lambda| \to \infty$ on the strips of width $2\tau > 0$,

$$\Lambda_\tau := \{ \lambda \in \mathbb{C}^* \ : \ |\mathrm{Im}\lambda| \le \tau, \ |\lambda| \ge 1 \}$$

as well as the asymptotics as $|\lambda| \to 0$ on

$$\Lambda_\tau^- := \{ \lambda \in \mathbb{C}^* \ : \ |\mathrm{Im}(16\lambda)^{-1}| \le \tau, \ |(16\lambda)^{-1}| \ge 1 \}.$$

The main ingredient are special solutions of equation (2.3), $\partial_x M = J(\lambda - A - B^2/\lambda)M$. First we consider the case $|\lambda| \to \infty$. It turns out to be more convenient to consider $\mathcal{M}$ introduced in (2.4), instead of M. More precisely, for any given potential $(q,p) \in H_c^{N+2}$, and $N \ge 0$, we look for solutions of equation (2.11),

$$\partial_x \mathcal{F} = -R(\lambda - \mathcal{A} - \mathcal{B}^2/\lambda)\mathcal{F} \tag{2.37}$$

of the form

$$\mathcal{F}_N(x,\lambda) = v_N(x,\lambda)\begin{pmatrix} 1 \\ \alpha_N(x,\lambda)\end{pmatrix} + \frac{1}{(2\mathrm{i}\lambda)^N}\mathcal{R}_N(x,\lambda)$$

where $|\lambda| \ge 1$, $\alpha_N(x,\lambda)$ is of the form

$$\alpha_N(x,\lambda) := \sum_{n=1}^{N} \frac{r_n(x)}{(2\mathrm{i}\lambda)^n},$$

$v_N(x,\lambda)$ is given by

$$v_N(x,\lambda) := \exp\Big(-\mathrm{i}\lambda x + \int_0^x \big(\alpha_N(t,\lambda)\big[\psi(t) - \frac{\mathrm{i}}{16\,\lambda}\sinh(q(t))\big] + \frac{\mathrm{i}}{16\,\lambda}\cosh(q(t))\big)\,\mathrm{d}t\Big)$$

with $\psi(x) = \frac{1}{4}(Pp(x) + \partial_x q(x))$ and the error term $\mathcal{R}_N(x,\lambda)$ satisfies the initial condition $\mathcal{R}_N(0,\lambda) = (0,0)$. If $\partial_x\mathcal{F}_N = -R(\lambda - \mathcal{A} - \mathcal{B}^2/\lambda)\mathcal{F}_N$ then $\mathcal{R}_N$ satisfies the following inhomogeneous equation

$$-\big(\partial_x + R(\lambda - \mathcal{A} - \mathcal{B}^2/\lambda)\big)\frac{\mathcal{R}_N}{(2\mathrm{i}\lambda)^N} = \mathcal{E}_N$$

where

$$\mathcal{E}_N(x,\lambda) := \big(\partial_x + R(\lambda - \mathcal{A} - \mathcal{B}^2/\lambda)\big)v_N(x,\lambda)\begin{pmatrix} 1 \\ \alpha_N(x,\lambda)\end{pmatrix}.$$

The function $v_N(x,\lambda)$ is chosen in such a way that the first component of $\mathcal{E}_N$ vanishes. Indeed, one has with $' = \partial_x$

$$\begin{aligned}\frac{1}{v_N}\partial_x\Big(v_N\begin{pmatrix} 1 \\ \alpha_N\end{pmatrix}\Big) =& \frac{v_N'}{v_N}\begin{pmatrix} 1 \\ \alpha_N\end{pmatrix} + \begin{pmatrix} 0 \\ \alpha_N'\end{pmatrix} \\ =& (-\mathrm{i}\lambda + \alpha_N\big[\psi - \frac{\mathrm{i}}{16\,\lambda}\sinh(q)\big] + \frac{\mathrm{i}}{16\,\lambda}\cosh(q))\begin{pmatrix} 1 \\ \alpha_N\end{pmatrix} + \begin{pmatrix} 0 \\ \alpha_N'\end{pmatrix}\end{aligned}$$

$$R\lambda\begin{pmatrix}1\\ \alpha_N\end{pmatrix}=\begin{pmatrix}\mathrm{i}\lambda\\ -\mathrm{i}\lambda\alpha_N\end{pmatrix}$$

$$-R\mathcal{A}\begin{pmatrix}1\\ \alpha_N\end{pmatrix}=\mathrm{i}\psi RJ\begin{pmatrix}1\\ \alpha_N\end{pmatrix}=-\psi\begin{pmatrix}\alpha_N\\ 1\end{pmatrix}$$

$$-R\mathcal{B}^2/\lambda\begin{pmatrix}1\\ \alpha_N\end{pmatrix}=-R\frac{1}{16\,\lambda}\begin{pmatrix}\cosh(q) & -\sinh(q)\\ -\sinh(q) & \cosh(q)\end{pmatrix}\begin{pmatrix}1\\ \alpha_N\end{pmatrix}=\frac{1}{16\,\lambda}\begin{pmatrix}\mathrm{i}\alpha_N\sinh(q)-\mathrm{i}\cosh(q)\\ \mathrm{i}\alpha_N\cosh(q)-\mathrm{i}\sinh(q)\end{pmatrix}.$$

Combining these computations yields $\frac{1}{v_N}\mathcal{E}_N=\begin{pmatrix}0\\ e_N\end{pmatrix}$ where

$$e_N:=\alpha_N'+\alpha_N\big(-2\mathrm{i}\lambda-\frac{1}{2\mathrm{i}\lambda}\frac{1}{4}\cosh(q)\big)+\alpha_N^2\big(\psi+\frac{1}{2\mathrm{i}\lambda}\frac{1}{8}\sinh(q)\big)-\psi+\frac{1}{2\mathrm{i}\lambda}\frac{1}{8}\sinh(q).$$

The coefficients r_n, $1\le n\le N$, of α_N are now determined in such a way that $e_N(x,\lambda)=O(\lambda^{-N})$. Substituting $\alpha_N=\sum_{n=1}^N\frac{r_n}{(2\mathrm{i}\lambda)^n}$ into the expression for e_N one obtains

$$\begin{aligned}e_N=&\sum_{n=1}^N\frac{r_n'}{(2\mathrm{i}\lambda)^n}-\sum_{n=1}^N\frac{r_n}{(2\mathrm{i}\lambda)^{n-1}}-\frac{1}{4}\cosh(q)\sum_{n=1}^N\frac{r_n}{(2\mathrm{i}\lambda)^{n+1}}-\psi+\frac{1}{2\mathrm{i}\lambda}\frac{1}{8}\sinh(q)\\&+\frac{1}{8}\sinh(q)\sum_{n=2}^N\Big(\sum_{k=1}^{n-1}r_kr_{n-k}\Big)\frac{1}{(2\mathrm{i}\lambda)^{n+1}}+\psi\sum_{n=2}^N\Big(\sum_{k=1}^{n-1}r_kr_{n-k}\Big)\frac{1}{(2\mathrm{i}\lambda)^n}.\end{aligned}$$

Hence the r_n can be determined recursively,

$$r_1:=-\psi,\quad \psi=\frac{1}{4}(Pp+q') \tag{2.38}$$

$$r_2:=r_1'+\frac{1}{8}\sinh(q)=-\psi'+\frac{1}{8}\sinh(q) \tag{2.39}$$

$$\begin{aligned}r_3:=&r_2'-\frac{1}{4}\cosh(q)r_1+\psi r_1^2\\=&-\psi''+\frac{1}{8}q'\cosh(q)+\frac{1}{4}\psi\cosh(q)+\psi^3\end{aligned} \tag{2.40}$$

and recursively, for $3\le n\le N-1$,

$$r_{n+1}:=r_n'-\frac{1}{4}r_{n-1}\cosh(q)+\psi\sum_{k=1}^{n-1}r_kr_{n-k}-\frac{1}{8}\sinh(q)\big(\sum_{k=1}^{n-2}r_kr_{n-1-k}\big). \tag{2.41}$$

As a consequence, $\alpha_N(x,\lambda)$ is in $H^1_{\mathbb{C}}$ for $(q,p)\in H_c^{N+2}$. Having determined the coefficients r_n, $1\le n\le N$, in this way it follows that $e_N=O(\lambda^{-N})$ as claimed. As a consequence,

$$-\big(\partial_x+R(\lambda-\mathcal{A}-\mathcal{B}^2/\lambda)\big)\mathcal{R}_N=v_N\begin{pmatrix}0\\ O(1)\end{pmatrix}.$$

By this method of the variation of constants one has, taking into account that $\mathcal{R}_N(0,\lambda)=(0,0)$,

$$\mathcal{R}_N(x,\lambda)=-\mathcal{M}(x,\lambda)\int_0^x\mathcal{M}^{-1}(t,\lambda)v_N(t,\lambda)\begin{pmatrix}0\\ O(1)\end{pmatrix}\mathrm{d}t.$$

Recall that by (2.4), $\mathcal{M}=TMT^{-1}$ where $T=\begin{pmatrix}1 & \mathrm{i}\\ 1 & -\mathrm{i}\end{pmatrix}$. Hence $\mathcal{M}^{-1}=TM^{-1}T^{-1}$ where in view of the Wronskian identity $M^{-1}=\begin{pmatrix}m_4 & -m_2\\ -m_3 & m_1\end{pmatrix}$. The estimates of Theorem 2.2 then shows that $v_N(x,\lambda)$ and $\mathcal{M},\mathcal{M}^{-1}$ are bounded on $[0,1]\times\Lambda_\tau$ uniformly on bounded subsets of H_c^{N+2}, implying that $\mathcal{R}_N(x,\lambda,q,p)=O(1)$ uniformly on $[0,1]\times\Lambda_\tau$ and uniformly on bounded subsets of H_c^{N+2}. Similarly, we construct solutions of (2.37) of the form

$$\mathcal{G}_N(x,\lambda)=w_N(x,\lambda)\begin{pmatrix}\beta_N(x,\lambda)\\ 1\end{pmatrix}+\frac{1}{(2\mathrm{i}\lambda)^N}\mathcal{S}_N(x,\lambda).$$

where $|\lambda| \geq 1$, $\beta_N(x,\lambda)$ is of the form

$$\beta_N(x,\lambda) := \sum_{n=1}^{N} \frac{s_n(x)}{(2\mathrm{i}\lambda)^n}$$

$w_N(x,\lambda)$ is given by

$$w_N(x,\lambda) := \exp\Big(\mathrm{i}\lambda x + \int_0^x \big(\beta_N(t,\lambda)\big[\psi(t) + \frac{\mathrm{i}}{16\,\lambda}\sinh(q(t))\big] - \frac{\mathrm{i}}{16\,\lambda}\cosh(q(t))\big)\,\mathrm{d}t\Big)$$

and the error term $\mathcal{S}_N(x,\lambda)$ satisfies the initial conditions $\mathcal{S}_N(x,\lambda) = (0,0)$. If $\partial_x \mathcal{G}_N = -R(\lambda - \mathcal{A} - \mathcal{B}^2/\lambda)\mathcal{G}_N$, then $\mathcal{S}_N$ satisfies the following inhomogeneous equation

$$-\big(\partial_x R(\lambda - \mathcal{A} - \mathcal{B}^2/\lambda)\big)\frac{\mathcal{S}_N}{(2\mathrm{i}\lambda)^N} = \mathcal{D}_N$$

where now

$$\mathcal{D}_N(x,\lambda) := \big(\partial_x + R(\lambda - \mathcal{A} - \mathcal{B}^2/\lambda)\big)w_N(x,\lambda)\begin{pmatrix}\beta_N(x,\lambda)\\ 1\end{pmatrix}.$$

The function $w_N(x,\lambda)$ is chosen in such a way that the second component of $\mathcal{D}_N$ vanishes. Indeed, one has

$$\begin{aligned}
\frac{1}{w_N}\partial_x\Big(w_N\begin{pmatrix}\beta_N\\ 1\end{pmatrix}\Big) =& \frac{w_N'}{w_N}\begin{pmatrix}\beta_N\\ 1\end{pmatrix} + \begin{pmatrix}\beta_N'\\ 0\end{pmatrix}\\
=& \big(\mathrm{i}\lambda + \beta_N(x,\lambda)\big[\psi(x) + \frac{\mathrm{i}}{16\,\lambda}\sinh(q(x))\big] - \frac{\mathrm{i}}{16\,\lambda}\cosh(q(x))\big)\begin{pmatrix}\beta_N\\ 1\end{pmatrix} + \begin{pmatrix}\beta_N'\\ 0\end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
R\lambda\begin{pmatrix}\beta_N\\ 1\end{pmatrix} =& \begin{pmatrix}\mathrm{i}\lambda\beta_N\\ -\mathrm{i}\lambda\end{pmatrix}\\
-R\mathcal{A}\begin{pmatrix}\beta_N\\ 1\end{pmatrix} =& R\mathrm{i}\psi J\begin{pmatrix}\beta_N\\ 1\end{pmatrix} = -\psi Z\begin{pmatrix}\beta_N\\ 1\end{pmatrix} = -\psi\begin{pmatrix}1\\ \beta_N\end{pmatrix}\\
-R\mathcal{B}^2/\lambda\begin{pmatrix}\beta_N\\ 1\end{pmatrix} =& -R\frac{1}{16\,\lambda}\begin{pmatrix}\cosh(q) & -\sinh(q)\\ -\sinh(q) & \cosh(q)\end{pmatrix}\begin{pmatrix}\beta_N\\ 1\end{pmatrix} = \frac{1}{16\,\lambda}\begin{pmatrix}-\mathrm{i}\beta_N\cosh(q) + \mathrm{i}\sinh(q)\\ -\mathrm{i}\beta_N\sinh(q) + \mathrm{i}\cosh(q)\end{pmatrix}.
\end{aligned}$$

Combining all this yields $\frac{1}{w_N}\mathcal{D}_N = \begin{pmatrix}d_N\\ 0\end{pmatrix}$ where

$$d_N := \beta_N' + \beta_N\big(2\mathrm{i}\lambda + \frac{1}{2\mathrm{i}\lambda}\frac{1}{4}\cosh(q)\big) + \beta_N^2\big(\psi - \frac{1}{2\mathrm{i}\lambda}\frac{1}{8}\sinh(q)\big) - \psi - \frac{1}{2\mathrm{i}\lambda}\frac{1}{8}\sinh(q).$$

The coefficients s_n, $1 \leq n \leq N$, of β_N are now determined in such a way that $d_N(x,\lambda) = O(\lambda^{-N})$. Substituting $\beta_N = \sum_{n=1}^N \frac{s_n}{(2\mathrm{i}\lambda)^n}$ into the expression for d_N one obtains

$$\begin{aligned}
d_N =& \sum_{n=1}^{N}\frac{s_N'}{(2\mathrm{i}\lambda)^n} + \sum_{n=1}^{N}\frac{s_n}{(2\mathrm{i}\lambda)^{n-1}} + \frac{1}{4}\cosh(q)\sum_{n=1}^{N}\frac{s_n}{(2\mathrm{i}\lambda)^{n+1}} - \psi - \frac{1}{2\mathrm{i}\lambda}\frac{1}{8}\sinh(q)\\
&+ \psi\sum_{n=2}^{N}\Big(\sum_{k=1}^{n-1}s_k s_{n-k}\Big)\frac{1}{(2\mathrm{i}\lambda)^n} - \frac{1}{8}\sinh(q)\sum_{n=2}^{N}\Big(\sum_{k=1}^{n-1}s_k s_{n-k}\Big)\frac{1}{(2\mathrm{i}\lambda)^{n+1}}.
\end{aligned}$$

Hence s_n can be determined recursively

$$\begin{aligned}
s_1 :=& \psi, \quad \psi = \frac{1}{4}(Pp + q')\\
s_2 :=& -s_1' + \frac{1}{8}\sinh(q) = -\psi' + \frac{1}{8}\sinh(q)\\
s_3 :=& -s_2' - s_1\frac{1}{4}\cosh(q) - \psi r_1^2 = \psi'' - \frac{1}{8}q'\cosh(q) - \psi\frac{1}{4}\cosh(q) - \psi^3
\end{aligned}$$

and for $n \geq 3$,

$$s_{n+1} = -s_n' - \frac{1}{4}\cosh(q)s_{n-1} - \psi\sum_{k=1}^{n-1} s_k s_{n-k} + \frac{1}{8}\sinh(q)\sum_{k=1}^{n-2} s_k s_{n-1-k}.$$

As a consequence, β_N is in $H^1_{\mathbb{C}}$ for $(q,p) \in H_c^{N+2}$, and $d_N = O(\lambda^{-N})$ as claimed. Arguing as for $\mathcal{R}_N$ one concludes that $\mathcal{S}_N(x,\lambda,q,p) = O(1)$ uniformly on $[0,1]\times\Lambda_\tau$ and uniformly on bounded subsets of H_c^{N+2}. The two solutions $\mathcal{F}_N$ and $\mathcal{G}_N$ turn out to be linearly independent for $\lambda \in \Lambda_\tau$ with $|\lambda|$ sufficiently large. Indeed, define the 2×2 matrix

$$\mathcal{M}_N(x,\lambda) := \big(\mathcal{F}_N(x,\lambda)\ \mathcal{G}_N(x,\lambda)\big)$$

with columns $\mathcal{F}_N(x,\lambda)$ and $\mathcal{G}_N(x,\lambda)$. Then $\mathcal{M}_N$ solves

$$\partial_x \mathcal{M}_N = -R(\lambda - \mathcal{A} - \mathcal{B}^2/\lambda)\mathcal{M}_N.$$

Since

$$\begin{aligned}\mathcal{F}_N(0,\lambda) =& v_N(0,\lambda)\begin{pmatrix}1\\ \alpha_N(0,\lambda)\end{pmatrix} + \frac{1}{(2\mathrm{i}\lambda)^N}\mathcal{R}_N(0,\lambda) = \begin{pmatrix}1\\ \alpha_N(0,\lambda)\end{pmatrix}\\ \mathcal{G}_N(0,\lambda) =& w_N(0,\lambda)\begin{pmatrix}\beta_N(0,\lambda)\\ 1\end{pmatrix} + \frac{1}{(2\mathrm{i}\lambda)^N}\mathcal{S}_N(0,\lambda) = \begin{pmatrix}\beta_N(0,\lambda)\\ 1\end{pmatrix}\end{aligned}$$

and by definition, $\alpha_N(0,\lambda), \beta_N(0,\lambda) = O(\lambda^{-1})$ it follows that $\mathcal{M}_N(0,\lambda) = \begin{pmatrix}1 & \beta_N(0,\lambda)\\ \alpha_N(0,\lambda) & 1\end{pmatrix}$ is invertible for $|\lambda|$ sufficiently large. By the uniqueness of the fundamental solution it follows that $\mathcal{M}(x,\lambda) = \mathcal{M}_N(x,\lambda)\mathcal{M}_N(0,\lambda)^{-1}$. Furthermore, since $\alpha_N(x,\lambda), \beta_N(x,\lambda) \in H^1_{\mathbb{C}}$ are 1-periodic in x one has

$$\mathcal{M}_N(1,\lambda) = \begin{pmatrix} v_N(1,\lambda) & w_N(1,\lambda)\beta_N(0,\lambda)\\ v_N(1,\lambda)\alpha_N(0,\lambda) & w_N(1,\lambda)\end{pmatrix} + O(\lambda^{-N}).$$

The Wronskian identity $\det\mathcal{M}_N(1,\lambda) = \det\mathcal{M}_N(0,\lambda)$ together with the asymptotics $\alpha_N(0,\lambda)\cdot\beta_N(0,\lambda) = O(\lambda^{-2})$ then implies that

$$1 - \alpha_N(0,\lambda)\beta_N(0,\lambda) = v_N(1,\lambda)w_N(1,\lambda)(1-\alpha_N(0,\lambda)\beta_N(0,\lambda)) + O(\lambda^{-N})$$

and in turn

$$v_N(1,\lambda)w_N(1,\lambda) = 1 + O(\lambda^{-N})$$

or

$$w_N(1,\lambda) = v_N(1,\lambda)^{-1} + O(\lambda^{-N}). \tag{2.42}$$

Furthermore $v_N(1,\lambda) = e^{\sigma_N(\lambda)} + O(\lambda^{-N})$ where $\sigma_N(\lambda) \equiv \sigma_N(\lambda,q,p)$ is given by

$$\begin{aligned}\sigma_N(\lambda) =& -\mathrm{i}\lambda + \sum_{n=1}^N \frac{1}{(2\mathrm{i}\lambda)^n}\int_0^1 r_n(x)\psi(x)\,\mathrm{d}x\\ &+\sum_{n=2}^N \frac{1}{(2\mathrm{i}\lambda)^n}\int_0^1 r_{n-1}(x)\frac{1}{8}\sinh(q(x))\,\mathrm{d}x - \frac{1}{2\mathrm{i}\lambda}\int_0^1\frac{1}{8}\cosh(q(x))\,\mathrm{d}x.\end{aligned}\tag{2.43}$$

Altogether, one then obtains the asymptotics

$$\mathcal{M}_N(1,\lambda) = \begin{pmatrix} e^{\sigma_N(\lambda)} & \beta_N(0)e^{-\sigma_N(\lambda)}\\ \alpha_N(0)e^{\sigma_N(\lambda)} & e^{-\sigma_N(\lambda)}\end{pmatrix} + O(\lambda^{-N}).$$

Since $\mathcal{M}_N(0,\lambda)^{-1} = \frac{1}{1-\alpha_N(0,\lambda)\beta_N(0,\lambda)}\begin{pmatrix}1 & -\beta_N(0,\lambda)\\ -\alpha_N(0,\lambda) & 1\end{pmatrix}$, the matrix $\mathcal{M}(1,\lambda) = \mathcal{M}_N(1,\lambda)\mathcal{M}_N(0,\lambda)^{-1}$ satisfies for $|\lambda|\to\infty$ the asymptotics

$$\frac{1}{1-\alpha_N(0,\lambda)\beta_N(0,\lambda)}\begin{pmatrix} e^{\sigma_N(\lambda)} - \alpha_N(0,\lambda)\beta_N(0,\lambda)e^{-\sigma_N(\lambda)} & -2\beta_N(0,\lambda)\sinh(\sigma_N(\lambda))\\ 2\alpha_N(0,\lambda)\sinh(\sigma_N(\lambda)) & e^{-\sigma_N(\lambda)} - \alpha_N(0,\lambda)\beta_N(0,\lambda)e^{\sigma_N(\lambda)}\end{pmatrix} + O(\lambda^{-N})$$

implying that

$$\Delta(\lambda) = \frac{1}{2}\mathrm{tr}\mathcal{M}(1,\lambda) = \frac{1}{2}\mathrm{tr}\big(\mathcal{M}_N(q,\lambda)\mathcal{M}_N(0,\lambda)^{-1}\big)$$

satisfies the asymptotes as $|\lambda| \to \infty$

$$\Delta(\lambda) = \cosh(\sigma_N(\lambda)) + O(\lambda^{-N}) \tag{2.44}$$

uniformly for $\lambda \in \Lambda_\tau$, $\tau > 0$ and uniformly on bounded subsets of H_c^{N+2}.

The equation (2.43) of $\sigma_N(\lambda)$ is written in the form

$$\sigma_N(\lambda) = -\mathrm{i}\lambda + \sum_{n=1}^{N} \frac{H_n}{(2\mathrm{i}\lambda)^n} \tag{2.45}$$

where $H_1, \dots, H_N$ are referred to as the first N Hamiltonians of the sinh-Gordon hierarchy. By (2.43) and (2.38)-(2.40)

$$H_1 = \int_0^1 -(\psi^2 + \frac{1}{8}\cosh(q))\,\mathrm{d}x \tag{2.46}$$

$$H_2 = \int_0^1 (-\psi'\psi + \psi\frac{1}{8}\sinh(q) - \psi\frac{1}{8}\sinh(q))\,\mathrm{d}x = 0 \tag{2.47}$$

$$H_3 = \int_0^1 \Big(\big(-\psi'' + \frac{1}{8}q'\cosh(q) + \frac{1}{4}\psi\cosh(q) + \psi^3\big)\psi + \big(-\psi' + \frac{1}{8}\sinh(q)\big)\frac{1}{8}\sinh(q)\Big)\,\mathrm{d}x \tag{2.48}$$

and recursively, for $3 \le n \le N$

$$H_n = \int_0^1 (r_n\psi + r_{n-1}\frac{1}{8}\sinh(q))\,\mathrm{d}x \tag{2.49}$$

with r_n, r_{n-1}, given by (2.39)-(2.41). Clearly, the Hamiltonians H_n are invariant under the sinh-Gordon flow. The identity $H_2 = 0$ is not a coincidence since it turns out that $\sigma_N(\lambda)$ is an odd function of λ:

***Lemma* 2.18** *For any $N \ge 1$ $\sigma_N(\lambda, q, p)$ is well defined on H_c^N and odd with respect to λ. It means that $\sigma_N(\lambda) = -\mathrm{i}\lambda - \mathrm{i}\sum_{1\le 2k+1\le N} \frac{(-1)^k H_{2k+1}}{(2\lambda)^{2k+1}}$.*

Proof. It follows from (2.45) and the definitions of H_n (2.46) - (2.49) and of r_n (2.38) - (2.41) that σ_N is well defined on H_c^N. To prove that $\sigma_N(\lambda)$ is odd in λ on H_c^N it suffices to consider $\sigma_N(\lambda)$ on H_c^{N+2}. By Lemma 2.14, $\Delta(\lambda) = \Delta(-\lambda)$ for any $\lambda \in \mathbb{C}^*$. Hence by the asymptotics (2.44),

$$\cosh(\sigma_N(-\lambda)) = \cosh(\sigma_N(\lambda)) + O(\lambda^{-N})$$

implying that $\sigma_N(-\lambda) = \sigma_N(\lambda) + O(\lambda^{-N})$ or $\sigma_N(-\lambda) = -\sigma_N(\lambda) + O(\lambda^{-N})$. Since $\sigma_N(\lambda) = -\mathrm{i}\lambda + \dots$ it then follows that

$$\sigma_N(-\lambda) = -\sigma_N(\lambda) + O(\lambda^{-N}).$$

□

Furthermore, since by Lemma 2.14(ii) (reciprocity), $\Delta(\frac{1}{16\,\lambda}, q, p) = \Delta(\lambda, -q, p)$, the asymptotics (2.44) for $\Delta(\lambda)$ as $|\lambda| \to \infty$ lead to corresponding asymptotics for $\lambda \in \Lambda_\tau^-$ as $|\lambda| \to 0$ where

$$\Lambda_\tau^- = \{\, \lambda \in \mathbb{C}^* : \ |\mathrm{Im}(16\lambda)^{-1}| \le \tau,\ |(16\lambda)^{-1}| \ge 1 \,\},$$

$$\Delta(\lambda) = \cosh(\sigma_N(\frac{1}{16\,\lambda}, -q, p)) + O(\lambda^N)$$

and

$$\sigma_{2m+1}(\frac{1}{16\,\lambda}, -q, p) = -\mathrm{i}\frac{1}{16\,\lambda} - \mathrm{i}\sum_{n=1}^{m}(-1)^n H_{2n+1}(-q,p)(8\lambda)^{2n+1}.$$

***Lemma* 2.19** *For any $m \ge 0$, H_{2m+1} extends to a real analytic function on H_c^{m+1}.*

Proof. By the definition (2.38) - (2.41), r_k is a polynomial of $\psi = \frac{1}{4}(Pp + q')$, $\cosh(q)$ and $\sinh(q)$ and their derivatives. More precisely each term in r_k has at most $k-1$ derivatives. Hence by partial integration it is possible to rewrite the integrand of

$$H_{2m+1} = \int_0^1 \big(r_{2m+1}\psi + r_{2m}\frac{1}{8}\sinh(q)\big)\,\mathrm{d}x$$

as a polynomial of ψ, $\cosh(q)$ and $\sinh(q)$ and their first m derivatives. Hence H_{2m+1} is well defined for any $(q,p) \in H_c^{m+1}$. □

Let us summarize the findings of this section in the following

***Theorem* 2.20** *For any $m \geq 0$ and $\tau > 0$, $\Delta(\lambda) \equiv \Delta(\lambda, q, p)$ admits asymptotic expansions*

$$\Delta(\lambda) = \cosh(\sigma_{2m+1}(\lambda, q, p)) + O(\lambda^{-(2m+1)}) \quad \textit{as} \quad |\lambda| \to \infty$$

$$\Big[\Delta(\lambda) = \cosh(\sigma_{2m+1}(\frac{1}{16\,\lambda}, -q, p)) + O(\lambda^{2m+1}) \quad \textit{as} \quad |\lambda| \to 0\Big]$$

The error terms are uniform for λ in Λ_τ $[\Lambda_\tau^-]$ and for bounded sets of potentials in H_c^{2m+3}. Furthermore $\sigma_{2m+1}(\lambda)$ is given by

$$\sigma_{2m+1}(\lambda) = -\mathrm{i}\lambda - \mathrm{i}\sum_{n=0}^{m} \frac{(-1)^n H_{2n+1}(q,p)}{(2\lambda)^{2n+1}}$$

$$\Big[\sigma_{2m+1}(\frac{1}{16\,\lambda}, -q, p) = -\mathrm{i}\frac{1}{16\,\lambda} - \mathrm{i}\sum_{n=1}^{m}(-1)^n H_{2n+1}(-q,p)(8\lambda)^{2n+1}.\Big]$$

where $H_1, H_3, \dots$ are the Hamiltonians in the sinh-*Gordon hierarchy introduced in* (2.46)-(2.49). *As a consequence of the Lax-pair formulation of the* sinh-*Gordon equation, $\Delta(\lambda), \lambda \in \mathbb{C}^*$, are integrals for this equation and so are in particular $H_n(q,p)$ and $H_n(-q,p)$. for any $1 \leq n \leq 2m+1$.*

Remark 2.21. Note that the sinh-Gordon Hamiltonian can be expressed in terms of $H_1(q,p)$ and $H_1(-q,p)$ by

$$H_{\sinh}(q,p) = \int_0^1 \frac{1}{2}((Pp)^2 + q_x^2) + \cosh(q)\,\mathrm{d}x = -4(H_1(q,p) + H_1(-q,p)). \tag{2.50}$$

Similarly, defining $H_*(q,p) := \int_0^1 (Pp)q_x\,\mathrm{d}x$ one has

$$H_*(q,p) = -4(H_1(q,p) - H_1(-q,p)). \tag{2.51}$$

Remark 2.22. Since by (2.42), $w_N(1,\lambda) = v_N(1,\lambda)^{-1} + O(\lambda^{-N})$ it follows from the definitions of $w_N(x,\lambda)$ and $v_N(x,\lambda)$ that

$$\begin{aligned}\mathrm{i}\lambda + &\int_0^1 \beta_N(t,\lambda)\big[\psi(t) - \frac{1}{2\mathrm{i}\lambda}\frac{1}{8}\sinh(q(t))\big] + \frac{1}{2\mathrm{i}\lambda}\frac{1}{8}\cosh(q(t)\,\mathrm{d}t \\ &= \mathrm{i}\lambda + \int_0^1 \alpha_N(t,\lambda)\big[-\psi(t) - \frac{1}{2\mathrm{i}\lambda}\frac{1}{8}\sinh(q(t))\big] + \frac{1}{2\mathrm{i}\lambda}\frac{1}{8}\cosh(q(t))\,\mathrm{d}t.\end{aligned}$$

Substituting the expressions for $\alpha_N(x,\lambda)$ and $\beta_N(x,\lambda)$ one obtains for $1 \leq n \leq N$.

$$\int_0^1 (r_n(x)\psi(x) + \frac{1}{8}\sinh(q(x))r_{n-1}(x))\,\mathrm{d}x = \int_0^1 -s_n(x)\psi(x) + \frac{1}{8}\sinh(q(x))s_{n-1}(x)\,\mathrm{d}x.$$

3 Spectra

The main purpose of this chapter is to study the asymptotics of the periodic and the Dirichlet spectrum of the operator $Q = Q_1\partial_x + Q_0$, introduced in (1.7). For our analysis it will be useful to introduce the domains $D_0 := \{\, z \in \mathbb{C} : \ |z - \frac{1}{4}| < \frac{1}{4\pi} \,\}$ and for any $n \geq 1$,

$$D_n := \{\, \lambda \in \mathbb{C} : \ |\lambda - n\pi| < \pi/3 \,\}, \quad D_{-n} := \{\, \lambda \in \mathbb{C} : \ \frac{1}{16\lambda} \in D_n \,\} \tag{3.1}$$

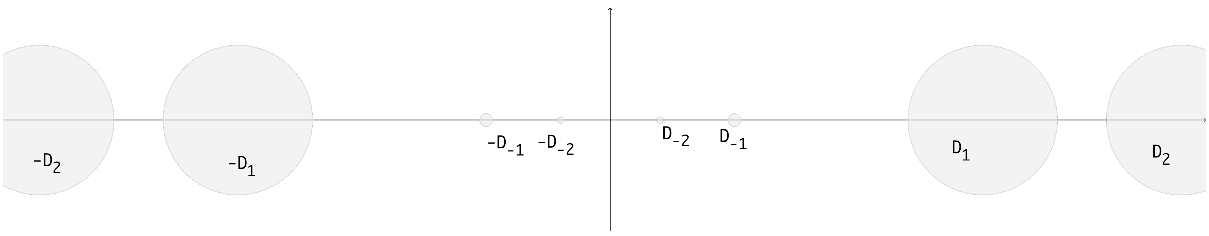

Figure 1: Illustration of the domains D_n, D_{-n}, $-D_{-n}$, $-D_n$ for $n = 1, 2$

Furthermore, let $B_0 := \{ \lambda \in \mathbb{C} : \ |\lambda| \le \pi/2 \}$ and for any $n \ge 1$

$$B_n := \{ \lambda \in \mathbb{C} : \ |\lambda| < n\pi + \pi/2 \}, \quad B_{-n} := \{ \lambda \in \mathbb{C} : \ |\lambda| \le \frac{1}{16(n\pi + \pi/2)} \}, \tag{3.2}$$

and denote by A_n the open annulus

$$A_n := B_n \setminus B_{-n}. \tag{3.3}$$

Recall by (2.32) that for any $v \in H^1_c$,

$$\grave{M}(\lambda, v) = M(x, \lambda, v)|_{x=1} \quad \text{and} \quad \grave{M}(\lambda, v) = \begin{pmatrix} \grave{m}_1 & \grave{m}_2 \\ \grave{m}_3 & \grave{m}_4 \end{pmatrix} \tag{3.4}$$

as well as $\grave{\mathcal{M}}(\lambda, v) = \mathcal{M}(x, \lambda, v)|_{x=1}$.

3.1 Dirichlet and Neumann spectrum

Denote by Q_{dir} the operator $Q = Q_1 \partial_x + Q_0$ with domain

$$H_{dir} := \{ F = (F_1, F_2, F_3, F_4) \in H^1([0,1], \mathbb{C}^4) : \ F_1(0) = F_1(1) = 0 \}.$$

Its spectrum is discrete and coincides with the Dirichlet spectrum of the spectral problem (2.2), defined as the set of eigenvalues with eigenfunctions $f = (f_1, f_2) \in H^1([0,1], \mathbb{C}^2)$ such that $f_1(0) = 0 = f_1(1)$. Clearly, for any $(q,p) \in H^1_c$, $\mu \in \mathbb{C}^*$ is a Dirichlet eigenvalue of (2.2) if there exists $a \in \mathbb{C}^*$ such that

$$\grave{M}(\mu, v) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = a \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{3.5}$$

We thus have the following

***Theorem* 3.1** *The Dirichlet spectrum of $Q(v)$ with $v \in H^1_c$ is the zero set of the function $\chi_D(\lambda) := \grave{m}_2(\lambda)$, $\{ \mu \in \mathbb{C}^* : \ \chi_D(\mu) = 0 \}$. Furthermore, the multiplicity $Mult(\mu, \chi_D)$ of a root μ of $\grave{m}_2$ equals to the algebraic multiplicity $Mult_a(\mu)$ of μ as a Dirichlet eigenvalue, defined as the dimension of the (finite dimensional) vector space $\bigcup_{n\ge1} \ker(\mu - Q_{dir}(v))^n$. The function χ_D is an analytic and compact function on $\mathbb{C}^* \times H^1_c$. For $v = 0$, $\chi_D(\lambda, 0) = \sin(\omega(\lambda))$.*

All the statements of Theorem 3.1 are shown in a straightforward way except the one on the multiplicity of the roots of χ_D. To prove it we first need to discuss some elementary properties of the Dirichlet eigenvalues and χ_D.

***Lemma* 3.2** *For any $(\lambda, v) \in \mathbb{C}^* \times H^1_c$*

(i) $\chi_D(-\lambda, v) = -\chi_D(\lambda, v)$, $\quad \chi_D(\frac{1}{16\lambda}, q, p) = -e^{-q(0)} \chi_D(\lambda, -q, p)$,

(ii) $\chi_D(\overline{\lambda}, \overline{v}) = \overline{\chi_D(\lambda, v)}$ *and* $\chi_D(\lambda, -v) = -\grave{m}_3(\lambda, v)$.

(iii) For $|\lambda| \to \infty$ with $\lambda \notin \bigcup_{n\ge1} D_n \cup (-D_n)$,

$$\chi_D(\lambda, v) = \chi_D(\lambda, 0)(1 + o(1))$$

locally uniformly in $v \in H^1_c$.

Proof. (i) Using that
$$\begin{pmatrix} \chi_D(\lambda,v) \\ 0 \end{pmatrix} = \begin{pmatrix} \grave{m}_2(\lambda,v) \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \grave{M}(\lambda,v) \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$
we obtain by Proposition 2.1
$$\begin{aligned}
\begin{pmatrix} \chi_D(-\lambda,v) \\ 0 \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \grave{M}(-\lambda,v) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} R\grave{M}(\lambda,v)R \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} -\grave{m}_2(\lambda,v) \\ 0 \end{pmatrix}. \\
\begin{pmatrix} \chi_D(\frac{1}{16\lambda},v) \\ 0 \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \grave{M}(\frac{1}{16\lambda},q,p) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} Re^{\mathrm{i}Rq(0)/2}\grave{M}(\lambda,-q,p)e^{-\mathrm{i}Rq(0)/2}R \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= -\begin{pmatrix} \mathrm{i}e^{-q(0)/2} & 0 \\ 0 & 0 \end{pmatrix} \grave{M}(\lambda,-q,p) \begin{pmatrix} 0 \\ -\mathrm{i}e^{-q(0)/2} \end{pmatrix} = -\begin{pmatrix} e^{-q(0)}\grave{m}_2(\lambda,-q,p) \\ 0 \end{pmatrix}.
\end{aligned}$$
(ii) is proved in a similar way as item (i).
(iii) By the same argument as in the proof of Lemma 2.17 one obtains the claimed asymptotics
$$\chi_D(\lambda,v) = \chi_D(\lambda,0)(1+o(1)).$$
□

As usual we denote by $\sqrt[+]{\lambda}$ the principal branch of the square root defined for λ in $\mathbb{C}\setminus(-\infty,0]$ and determined by $\sqrt[+]{1}=1$.

Lemma 3.3 *The Dirichlet eigenvalues at $v=0$ are $\frac{1}{4}, -\frac{1}{4}$ and*
$$\frac{n\pi}{2}\Big(\sqrt[+]{1+\frac{1}{4n^2\pi^2}}+1\Big), \quad \frac{n\pi}{2}\Big(\sqrt[+]{1+\frac{1}{4n^2\pi^2}}-1\Big), \quad n\neq 0,$$
each eigenvalue having multiplicity one.

A first rough localization of the Dirichlet eigenvalues is provided by the following

Lemma 3.4 (Counting Lemma) *For each potential in H_c^1 there exist a neighborhood U in H_c^1 and an integer $N>0$ so that for any $v\in U$, the function $\lambda\mapsto\chi_D(\lambda,v)$ has exactly one root in each of the domains $D_n, -D_n, D_{-n}, -D_{-n}$ for any $n>N$ and exactly $2+4N$ in the annulus A_N, counted with their multiplicities. There are no other roots.*

Proof. By Lemma 3.2, for $|\lambda|\to\infty$ with $\lambda\notin\bigcup_{n\geq 1} D_n\cup(-D_n)$
$$\chi_D(\lambda,v) = \chi_D(\lambda,0)\,(1+o(1))$$
locally uniformly in v. Hence, for any potential in H_c^1 there is a neighborhood U and an integer $N\geq 1$ such that for any $v\in U$
$$|\chi_D(\lambda,v)-\chi_D(\lambda,0)| < |\chi_D(\lambda,0)| \tag{3.6}$$
$$|\chi_D(\lambda,-q,p)-\chi_D(\lambda,0,0)| < |\chi_D(\lambda,0,0)| \tag{3.7}$$
on the boundaries of the discs $D_n, -D_n$, and B_n for any $n\geq N$. It follows by Rouché's theorem that $\chi_D(\cdot,v)$ has as many roots inside any of the discs $\pm D_n$, $n\geq N$, as $\chi_D(\cdot,0)$. There are no other roots in $\mathbb{C}^*\setminus\Big(B_N\bigcup_{n\geq N}(D_n\cup -D_n)\Big)$. By Lemma 3.2(i)
$$|e^{q(0)}\chi_D(\frac{1}{16\lambda},q,p)-\chi_D(\frac{1}{16\lambda},0,0)| = |\chi_D(\lambda,-q,p)-\chi_D(\lambda,0,0)| < |\chi_D(\lambda,0,0)|. \tag{3.8}$$
Since $\chi_D(\cdot,v)$ has the same roots as $e^{q(0)}\chi_D(\cdot,v)$ and $\chi_D(\frac{1}{16\lambda},0)=\chi_D(\lambda,0)$ it follows that $\chi_D(\lambda,v)$ has as many roots as $\chi_D(\lambda,0)$ inside any of the discs $D_{-n}, -D_{-n}$ with $n>N$. It remains to count the roots inside A_n with $n\geq N$. In order to apply Rouché's theorem we need to estimate χ_D on the

boundary of B_{-n}. Arguing as above one concludes that for any λ on the boundary of B_n with $n \geq N$ and $t, t+s \in [0,1]$,

$$\begin{aligned}
&|e^{(t+s)q(0)}\chi_D(\frac{1}{16\lambda},(t+s)v) - e^{tq(0)}\chi_D(\frac{1}{16\lambda},tv)|\\
=&|\chi_D(\lambda,-(t+s)v) - \chi_D(\lambda,-tv)|\\
<&\frac{1}{2}|\chi_D(\lambda,-tv)| = \frac{1}{2}|e^{tq(0)}\chi_D(\frac{1}{16\lambda},tv)|.
\end{aligned}$$

After division by $|e^{tq(0)}|$ one gets

$$|e^{sq(0)}\chi_D(\frac{1}{16\lambda},(t+s)v) - \chi_D(\frac{1}{16\lambda},tv)| \leq \frac{1}{2}|\chi_D(\frac{1}{16\lambda},tv)|.$$

Chose $\epsilon > 0$ such that

$$|e^{sq(0)} - 1||\chi_D(\frac{1}{16\lambda},(t+s)v)| < \frac{1}{2}|\chi_D(\frac{1}{16\lambda},tv)|$$

for λ on the boundary of B_n with $n \geq N$, $t \in [0,1]$, and $0 \leq s < \epsilon$. Then

$$|\chi_D(\lambda,(t+s)v) - \chi_D(\lambda,tv)| < |\chi_D(\lambda,tv)|$$

on the boundary of A_n. By Rouché's Theorem it then follows that the number of roots of $\chi_D(\cdot,tv)$ inside anyA_n is independent of $t \in [0,1]$. Since $\chi_D(\cdot,0)$ has $2+4N$ roots inside A_N so does $\chi_D(\cdot,v)$. Furthermore, since $(A_n)_{n\geq N}$ is a covering of $\mathbb{C}^*$, there are no roots in $\mathbb{C}^*\setminus\Big(A_N \cup \bigcup_{n\geq N} D_n \cup (-D_n)\Big)$. □

Since by Lemma 3.2(i) $\chi_D(-\lambda,v) = -\chi_D(\lambda,v)$, it is enough to consider the Dirichlet eigenvalues of $Q(v)$ in

$$\mathbb{C}^+ := \{\ \lambda \in \mathbb{C}:\ \mathrm{Re}\lambda > 0\ \} \cup \mathrm{i}\mathbb{R}_{>0}. \tag{3.9}$$

For any $v \in H^1_c$, these eigenvalues, when counted with their multiplicities $Mult_a(\mu)$, can be listed as a bi-infinite sequence

$$0 \preceq \cdots \preceq \mu_{-2} \preceq \mu_{-1} \preceq \mu_0 \preceq \mu_1 \preceq \mu_2 \preceq \cdots. \tag{3.10}$$

Here $\preceq$ is the ordering of complex numbers in $\mathbb{C}^+$ defined as follows: for $a, b \in \mathbb{C}^+$, $a \preceq b$,

$$\Big[|a| < |b|\Big] \quad \text{or} \quad \Big[|a| = |b| \quad \text{and} \quad \mathrm{Im}a \leq \mathrm{Im}b\Big]. \tag{3.11}$$

Note that $\preceq$ is a total ordering of $\mathbb{C}^+$. One of its feature is that for any $a \in \mathbb{C}^+$, $a \preceq \mathrm{i}|a|$. In particular, ordering the Dirichlet eigenvalues in this way one has that $\mu_n = n\pi + o(1)$ and $\frac{1}{16\mu_{-n}} = n\pi + o(1)$.

Proof of Theorem 3.1. Recall that $\mu \in \mathbb{C}^*$ is a Dirichlet eigenvalue of $Q(v)$ iff there exists $a \in \mathbb{C}^*$ and $F \in H^1_{loc}(\mathbb{R},\mathbb{C}^4)$ with $QF = \mu F$ and

$$F(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{e^{q(0)/2}}{4\,\lambda} \end{pmatrix}, \quad F(1) = a\begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{e^{q(0)/2}}{4\,\lambda} \end{pmatrix}.$$

One concludes that the geometric multiplicity of μ is one. The algebraic multiplicity $Mult_a(\mu)$ of μ equals the dimension of the range of the Riesz projector

$$\Pi(\mu) := \frac{1}{2\pi\mathrm{i}}\int_{\Gamma(\mu)} (\lambda - Q_{dir}(v))^{-1}\,\mathrm{d}\lambda,$$

where $\Gamma(\mu)$ is a counterclockwise oriented contour around μ so that all Dirichlet eigenvalues of $Q(v)$ except μ are outside of $\Gamma(\mu)$. Since $(\lambda - Q_{dir}(v))^{-1}$ is a compact operator $Mult_a(\mu)$ is finite and $Mult_a(\mu) = \mathrm{tr}\Pi(\mu)$. □

***Lemma* 3.5** *For any Dirichlet eigenvalue μ of $Q(v)$ with $v \in H^1_c$, $Mult_a(\mu) = Mult(\mu,\chi_D)$.*

Proof. By Proposition 2.1 and Lemma 3.2, $Mult_a(\mu_n) = Mult_a(-\mu_n)$ and $Mult(\mu_n, \chi_D) = Mult(-\mu_n, \chi_D)$. Hence it suffices to consider the Dirichlet eigenvalues in $\mathbb{C}^+$. By Lemma 3.3, the Dirichlet eigenvalues at $v = 0$ contained in $\mathbb{C}^+$ are given by $\mu_k^0 = \frac{1}{2}\left(k\pi + \sqrt{k^2\pi^2 + 1/4}\right)$, $k \in \mathbb{Z}$, and since $\chi_D(\lambda, 0) = \sin(\omega(\lambda))$ one has $Mult(\mu_k^0, \chi_D) = 1$. Note that for any $k \in \mathbb{Z}$

$$\big(\sin(\omega_k^0 x), \cos(\omega_k^0 x), \frac{1}{4\,\mu_k^0}\sin(\omega_k^0 x), \frac{1}{4\,\mu_k^0}\cos(\omega_k^0 x)\big)$$

is an eigenfunction of $Q(0)$, corresponding to the eigenvalue μ_k^0, where $\omega_k^0 \equiv \omega(\mu_k^0) = \pi|k|$. Since $Q_{dir}(0)$ is selfadjoint with respect to the canonical inner product on $L^2([0,1], \mathbb{C}^4)$, the algebraic multiplicity $Mult_a(\mu_k^0)$ is one. Since $Mult(\mu_k^0, \chi_D) = 1$ it then follows that $Mult_a(\mu_k^0) = Mult(\mu_k^0, \chi_D)$ for any $k \in \mathbb{Z}$. Now let $v_0 \in H_c^1$ and consider the line segment $[0, v_0]$, from 0 to v_0 in H_c^1. Since it is compact it follows by the Counting Lemma that there exist a neighborhood U of $[0, v_0]$ in H_c^1 and $N \geq 1$ such that for any potential v in U and $|k| > N$, $\mu_k(v) \in D_k$. It implies that $Mult(\mu_k, \chi_D) = 1$. Choosing $\Gamma(\mu_n) := \partial D_n$ one also sees that $Mult_a(\mu_k) = 1$ for any $|k| > N$. For the remaining $4N + 2$ Dirichlet eigenvalues in A_N consider the Riesz projector

$$\Pi_N(v) := \frac{1}{2\pi \mathrm{i}} \int_{\partial A_N} (\lambda - Q_{dir}(v))^{-1}\, \mathrm{d}\lambda.$$

Denote by $\mathcal{R}_N(v)$ the range of $\Pi_N(v)$ and let $\Lambda_N(v) = Q(v)|_{\mathcal{R}_N}$. Since $\mathrm{tr}\Pi_N$ is continuous and hence constant in U, the dimension of $\mathcal{R}_N(v)$ is $4N + 2$ and Λ_N maps $\mathcal{R}_N$ onto itself. Thus,

$$\xi_N(\lambda, v) := \det\left(\lambda - \Lambda_N(v)\right)$$

is a polynomial of degree $4N + 2$. By construction, its roots are precisely the Dirichlet eigenvalues inside A_N, counted with their algebraic multiplicities. On the other hand, consider the polynomial

$$\zeta_N(\lambda, v) := \prod_{|k| \leq N} (\lambda - \mu_k(v))(\lambda + \mu_k(v))$$

formed by the roots $\mu_k(v), -\mu_k(v)$, $|k| \leq N$, of χ_D inside A_N counted with their multiplicities $Mult(\mu_k, \chi_D)$. By the analyticity of χ_D and the argument principle, the coefficients of ζ_N are in fact analytic functions in $v \in U$. The same is true for the coefficients of ξ_N.

Note that by the same argument as in Lemma 3.4 there is a neighborhood $U^{(0)}$ of 0 in H_c^1 so that on $U^{(0)}$, $\mu_k \in D_k$ for any $k \in \mathbb{Z}$. Hence ξ_N and ζ_N coincide on $U^{(0)} \cap U$ $(\neq \emptyset)$. By the analyticity of the coefficients of ζ_N and ξ_N we conclude that $\xi_N(\cdot, v) = \zeta_N(\cdot, v)$ for all $v \in U^{(0)} \cap U$, implying that on U, $Mult_a(\mu_k(v)) = Mult(\mu_k(v), \chi_D)$ for any $|k| \leq N$. □

Denote by Q_{neu} the operator $Q(v)$ with domain

$$H_{neu} := \{\ F = (F_1, F_2, F_3, F_4) \in H^1([0,1], \mathbb{C}^4) :\ F_2(0) = 0 = F_2(1)\ \}.$$

Its spectrum, referred to as Neumann spectrum, is discrete and coincides with the Neumann spectrum of the spectral problem (2.2), defined as the set of eigenvalues with eigenfunctions $f = (f_1, f_2) \in H^1([0,1], \mathbb{C}^2)$ such that $f_2(0) = 0 = f_2(1)$. Clearly, for any $v \in H_c^1$, $\nu \in \mathbb{C}^*$ is an eigenvalue of (2.2) if there exists $a \in \mathbb{C}^*$ such that

$$\grave{M}(\lambda, v)\begin{pmatrix}1\\0\end{pmatrix} = a\begin{pmatrix}1\\0\end{pmatrix}.$$

Since by Lemma 3.2(ii) $\chi_N(\lambda, v) = -\chi_D(\lambda, -v)$, Theorem 3.1 and Lemma 3.4 yield the following results.

Theorem 3.6 *The Neumann spectrum of $Q(v)$ with $v \in H_c^1$ is the zero set of the function $\chi_N(\lambda) := \grave{m}_3(\lambda)$, $\{\ \nu \in \mathbb{C}^* :\ \chi_N(\nu) = 0\ \}$. Furthermore, the multiplicity $Mult(\nu, \chi_N)$ of the root ν equals the algebraic multiplicity $Mult_a(\nu)$ of ν as a Neumann eigenvalue, i.e. to the dimension of the (finite dimensional) vector space $\bigcup_{n\geq 1}\ker(\nu - Q_{neu}(v))^n$. The function χ_N is antisymmetric in λ and hence the Neumann spectrum is even in λ. The function χ_N is analytic and compact on $\mathbb{C}^* \times H_c^1$. For $v = 0$, $\chi_N(\lambda, 0) = -\sin(\omega(\lambda))$. Finally, results corresponding to Lemma 3.4 also hold for the Neumann eigenvalues.*

Lemma 3.3 and Lemma 3.2(ii) lead to the following

***Lemma* 3.7** *The Neumann spectrum of $Q(v)$ at $v = 0$ coincides with the Dirichlet spectrum of $Q(v)$ at $v = 0$.*

The Neumann eigenvalues of $Q(v)$ at $v \in H^1_c$, contained in $\mathbb{C}^+$ and counted with their algebraic multiplicities can be listed as a bi-infinite sequence

$$0 \preceq \cdots \preceq \nu_{-2} \preceq \nu_{-1} \preceq \nu_0 \preceq \nu_1 \preceq \nu_2 \preceq \cdots \tag{3.12}$$

so that for $|k|$ sufficiently large, ν_k is the unique Neumann eigenvalue of $Q(v)$ in the disc D_k.

We finish this section with the following useful identity.

***Lemma* 3.8** *For any Dirichlet or Neumann eigenvalue λ of $Q(v)$ with $v \in H^1_c$,*

$$\Delta^2(\lambda, v) - 1 = \delta^2(\lambda, v).$$

Proof. By the Wronskain identity $1 = \grave{m}_1\grave{m}_4 - \grave{m}_2\grave{m}_3$. Hence

$$\begin{aligned}\Delta^2 - 1 =& \frac{1}{4}(\grave{m}_1 + \grave{m}_4)^2 - 1\\ =& \frac{1}{4}(\grave{m}_1 + \grave{m}_4)^2 - \grave{m}_1\grave{m}_4 + \grave{m}_2\grave{m}_3\\ =& \frac{1}{4}(\grave{m}_1 - \grave{m}_4)^2 + \grave{m}_2\grave{m}_3 = \delta^2 + \grave{m}_2\grave{m}_3.\end{aligned}$$

Since the Dirichlet and Neumann eigenvalues are roots of $\grave{m}_2\grave{m}_3$ the claimed identity follows. □

3.2 Periodic spectrum

In this section we describe the periodic spectrum $spec_{per}(Q)$ of the operator $Q = Q_1\partial_x + Q_0$ with domain given by the subspace of functions F in

$$H_{per\pm} := \{\ F \in H^1_{loc}(\mathbb{R}, \mathbb{C}^4) :\ F(x+1) = \pm F(x)\ \ \forall\ x \in \mathbb{R}\ \}.$$

It coincides with the periodic spectrum of the spectral problem (2.2). Hence a complex number $\lambda \in \mathbb{C}^*$ is in $spec_{per}(Q)$ iff $\grave{M}(\lambda, v)$ has an eigenvalue ± 1. Since $\det(\grave{M}) = 1$, the eigenvalues $\xi_\pm$ of $\grave{M}(\lambda) \equiv \grave{M}(\lambda, v)$ satisfy

$$0 = \det(\xi_\pm I - \grave{M}(\lambda)) = \xi_\pm^2 - 2\Delta(\lambda)\xi_\pm + 1, \tag{3.13}$$

and thus are given by

$$\xi_\pm = \Delta(\lambda) \pm \sqrt{\Delta^2(\lambda) - 1}. \tag{3.14}$$

Note that in (3.14) ξ_+ and ξ_- are determined up to the choice of a branch of $\sqrt{\Delta^2(\lambda) - 1}$.

***Theorem* 3.9** *The periodic spectrum of $Q(v)$ with $v \in H^1_c$ is discrete and coincides with the zero set $\{\ \lambda \in \mathbb{C}^* :\ \chi_p(\lambda, v) = 0\ \}$ of the function*

$$\chi_p(\lambda, v) := \Delta^2(\lambda, v) - 1.$$

Furthermore, the multiplicity of any root of χ_p coincides with its algebraic multiplicity as a periodic eigenvalue. By Lemma 2.14(i) and (ii) the periodic spectrum is invariant under the involution $\lambda \to -\lambda$ and for any periodic eigenvalue of $Q(q, p)$, $\frac{1}{16\lambda}$ is a periodic eigenvalue of $Q(-q, p)$.

Proof. Let $v \in H^1_c$ by given. By (2.2) for any $\lambda \in \mathbb{C}^*$ and $F \in H^1_{loc}(\mathbb{R}, \mathbb{C}^4)$, the identity $QF = \lambda F$ is equivalent to $F = (f, \lambda^{-1}Bf)$ where $f(x) = M(x, \lambda, v)f(0)$. Hence the existence of a solution F of $QF = \lambda F$ with $F(1) = \pm F(0)$ is equivalent to ± 1 being an eigenvalue of $M(1, \lambda, v)$. By (3.14), ± 1 is an eigenvalue of $M(1, \lambda, v)$ iff $\Delta(\lambda, v) = \pm 1$. This proves the characterization.

The statement on the algebraic multiplicity of periodic eigenvalues is proved as the corresponding one for the Dirichlet eigenvalues of (c.f. Lemma 3.5) and hence we omit its proof. □

For $v = 0$ the periodic spectrum of $Q(v)$ can be computed explicitly. By Lemma 2.16, $\chi_p(\lambda, 0) = \cos^2(\omega(\lambda)) - 1 = -\sin^2(\omega(\lambda))$ where we recall that $\omega(\lambda) = \lambda - \frac{1}{16\lambda}$.

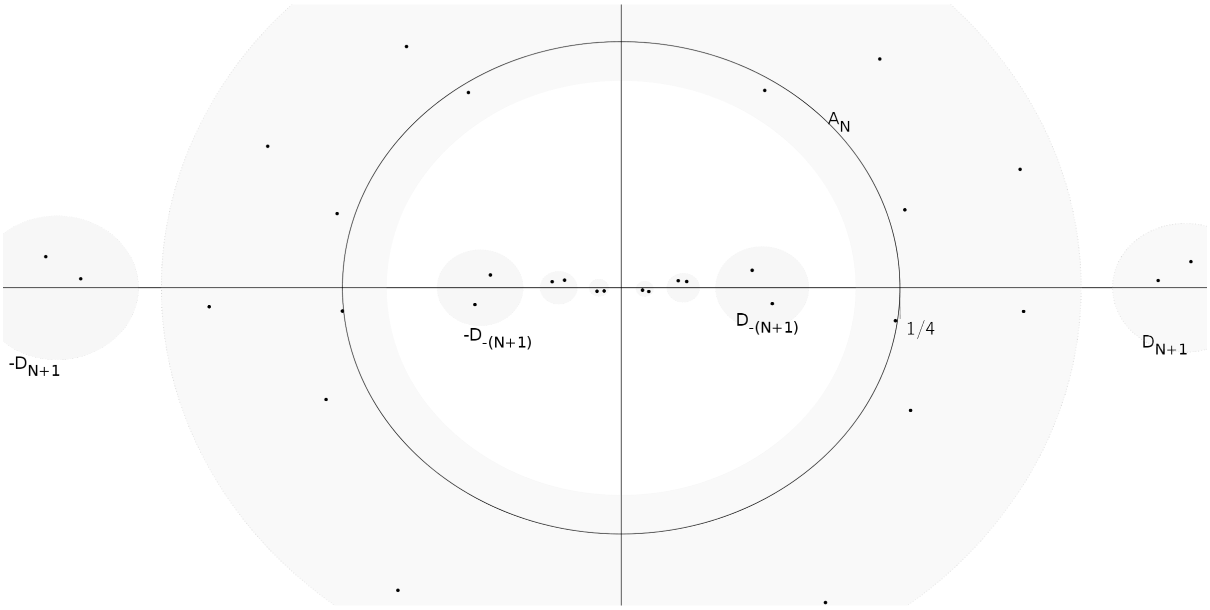

Figure 2: Localization of periodic eigenvalues

***Corollary* 3.10** *The periodic eigenvalues of $Q(v)$ for $v = 0$ are*

$$\{ \frac{n\pi}{2}\Big(\sqrt[+]{1+\frac{1}{4n^2\pi^2}}+1\Big), \quad \frac{n\pi}{2}\Big(\sqrt[+]{1+\frac{1}{4n^2\pi^2}}-1\Big) \;:\; n \neq 0 \,\} \cup \{\frac{1}{4}, -\frac{1}{4}\}.$$

Each eigenvalue has algebraic multiplicity two.

It is convenient to list the two sequences of periodic eigenvalues of $Q(0)$ with their algebraic multiplicities as follows

$$0 < \cdots < \lambda_{-1}^- = \lambda_{-1}^+ < \lambda_0^- = \lambda_0^+ = \frac{1}{4} < \lambda_1^- = \lambda_1^+ < \lambda_2^- = \lambda_2^+ < \cdots$$

$$\cdots < -\lambda_1^- = -\lambda_1^+ < -\lambda_0^- = -\lambda_0^+ = -\frac{1}{4} < -\lambda_{-1}^- = -\lambda_{-1}^+ < -\lambda_{-2}^- = -\lambda_{-2}^+ < \cdots < 0.$$

We note that

$$\Delta(\lambda_k^+, 0) = \Delta(\lambda_k^-, 0) = (-1)^k, \quad \forall\, k \in \mathbb{Z}, \quad \text{and} \quad \lambda_{-k}^+ = \frac{1}{16\,\lambda_k^+} \;\; \forall\, k \geq 0. \tag{3.15}$$

The periodic spectrum of $Q(v)$ for arbitrary $v \in H_c^1$ is asymptotically close to the one of $Q(0)$. Recall that the domains D_n and D_{-n}, $n \geq 1$ are defined in (3.1).

***Lemma* 3.11** (Counting Lemma) *For each potential in H_c^1 there exist a neighborhood U in H_c^1 and an integer $N > 0$, such that for every $v \in U$, the entire function $\chi_p(\lambda, v)$ has exactly two roots in each of the domains D_n, $-D_n$, D_{-n}, and $-D_{-n}$ with $n > N$ and exactly $4 + 8N$ roots in the annulus A_N, counted with their multiplicities. There are no further roots.*

Proof. By Lemma 2.17,

$$\chi_p(\lambda, v) = \chi_p(\lambda, 0)\,(1 + o(1))$$

for $|\lambda| \to \infty$ with $\lambda \notin \bigcup_{n\geq 1} D_n \cup (-D_n)$, locally uniformly in $v \in H_c^1$. Hence, for any potential in H_c^1 there is a neighborhood U and an integer $N \geq 1$ such that for any $v \in U$

$$|\chi_p(\lambda, v) - \chi_p(\lambda, 0)| < |\chi_p(\lambda, 0)| \tag{3.16}$$

$$|\chi_p(\lambda, -q, p) - \chi_p(\lambda, 0, 0)| < |\chi_p(\lambda, 0, 0)| \tag{3.17}$$

on the boundaries of the discs $D_n, -D_n$, and B_n (defined in (3.2)) for any $n \geq N$. The estimate (3.17) implies by Lemma 2.14 that

$$|\chi_p(\frac{1}{16\lambda}, q, p) - \chi_p(\frac{1}{16\lambda}, 0, 0)| = |\chi_p(\lambda, -q, p) - \chi_p(\lambda, 0, 0)| < |\chi_p(\lambda, 0, 0)| = |\chi_p(\frac{1}{16\lambda}, 0, 0)|.$$

It then follows that for any $n \geq N$, (3.16) holds on the boundaries of $\pm D_n, \pm D_{-n}$ and B_n, B_{-n} and hence also on the boundary of $A_n = B_n \setminus B_{-n}$. Therefore by Rouché's theorem, $\chi_p(\cdot, v)$ has as many roots inside any of the discs $\pm D_n, \pm D_{-n}$ and annuli A_n as $\chi_p(\lambda, 0)$ for any $n \geq N$. Since $(A_n)_{n \geq N}$ is a covering of $\mathbb{C}^*$ the same argument shows that $\chi_p(\lambda, v)$ has no roots in $\mathbb{C}^* \setminus \big(A_N \cup \bigcup_{n>N}(D_n \cup (-D_n) \cup D_{-n} \cup (-D_{-n}))\big)$.
□

Note that if $v \in H^2_c$ one can estimate the bound N of Lemma 3.11 in terms of $\|v\|_2$. Furthermore, by Lemma 2.14(i) (Reflection in λ) it is enough to consider the part of the periodic spectrum of $Q(v)$ in the half plane $\mathbb{C}^+$ (definition (3.9)) and by Lemma 3.11 (Counting Lemma) the periodic eigenvalues in $\mathbb{C}^+$, counted with their algebraic multiplicities, can be listed as a bi-infinite sequence

$$0 \preceq \cdots \preceq \lambda^-_{-1} \preceq \lambda^+_{-1} \preceq \lambda^-_0 \preceq \lambda^+_0 \preceq \lambda^-_1 \preceq \lambda^+_1 \preceq \cdots . \tag{3.18}$$

Note that the segment $\{\, tv \in H^1_c \;:\; t \in [0,1] \,\}$ connecting v to 0 in H^1_c is compact and hence the integer N of Lemma 3.11 can be chosen uniformly in $0 \leq t \leq 1$. Furthermore, for any $|k| \geq N$ $\Delta(\lambda^+_k(tv), tv) = \Delta(\lambda^-_k(tv), tv)$ and its sign is constant in t. We conclude that for such k , $\Delta(\lambda^\pm_k, v) = (-1)^k$. Such an identity does not hold for the remaining finitely many eigenvalues, unless v satisfies further conditions such as being (almost) real valued- see Section 6.1 and Section 6.2 for details.

We finish this section by a discussion on the roots of $\dot\Delta(\lambda, v) \equiv \partial_\lambda \Delta(\lambda, v)$. Since Δ is even with respect to the variable λ, $\dot\Delta$ is odd and hence it is again enough to look at the roots of $\dot\Delta$ in $\mathbb{C}^+$. For $v = 0$ one has $\Delta(\lambda) \equiv \Delta(\lambda, 0) = \cos(\omega(\lambda))$, where $\omega(\lambda) = \lambda - \frac{1}{16\lambda}$, and hence

$$\dot\Delta(\lambda) \equiv \dot\Delta(\lambda, 0) = -(1 + \frac{1}{16\lambda^2}) \sin(\omega(\lambda)). \tag{3.19}$$

The roots of $\dot\Delta(\lambda)$ in $\mathbb{C}^+$ are given by the set of complex numbers consisting of the bi-infinite sequence

$$\dot\lambda_k \equiv \dot\lambda_k(0) = \lambda^+_k(0), \quad \forall\, k \in \mathbb{Z}$$

and the additional root $\dot\lambda_* = \frac{i}{4}$. Each of these roots has multiplicity one.

By Lemma 2.14(ii) one has $-\frac{1}{16\lambda^2}\dot\Delta(\frac{1}{16\lambda}, q, p) = \dot\Delta(\lambda, -q, p)$. Since $-\frac{1}{16\lambda^2}\dot\Delta(\frac{1}{16\lambda}, q, p)$ and $\dot\Delta(\frac{1}{16\lambda}, q, p)$ have the same roots in $\mathbb{C}^*$ (counted with their multiplicities), we can use the same arguments as for the periodic eigenvalues of $Q(v)$, to prove the following:

***Lemma* 3.12** (Counting Lemma) *Given any potential in H^1_c there exists a neighborhood U of it in H^1_c and $N > 0$ (U and N can be chosen as in Lemma 3.11) so that for any $v \in U$, the function $\lambda \mapsto \dot\Delta(\lambda, v)$ has exactly one root in each of the domains D_n, $-D_n$, D_{-n}, and $-D_{-n}$ with $n > N$ and $4 + 4N$ roots in the annulus A_N. There are no other roots.*

By this lemma the roots of $\dot\Delta(\cdot, v)$ in $\mathbb{C}^+ \setminus A_N$, counted with their algebraic multiplicities, can be listed as a bi-infinite sequence

$$0 \preceq \cdots \preceq \dot\lambda_{-N-2} \preceq \dot\lambda_{-N-1} \preceq \dot\lambda_{N+1} \preceq \dot\lambda_{N+2} \preceq \cdots , \quad \dot\lambda_k \in D_k \;\; \forall\, |k| > N \tag{3.20}$$

such that any remaining root $\dot\lambda$ in $\mathbb{C}^+$ satisfies $\dot\lambda_{-N-1} \preceq \dot\lambda \preceq \dot\lambda_{N+1}$. It turns out that for arbitrary $v \in H^1_c$, these remaining roots cannot be listed in a way useful for our purposes. But in case v is (almost) real valued such a listing is possible – see Section 6.1 and 6.2 for details.

3.3 Estimates

The main purpose of this section is to establish estimates for the periodic, Dirichlet, and Neumann eigenvalues of the operator $Q(v)$. A first result concerns a priori bounds of the imaginary part of any of these eigenvalues.

***Lemma* 3.13** *For any $v \in H^2_c$ and any periodic, Dirichlet, or Neumann eigenvalue $\lambda \in \mathbb{C}^+$,*

$$|\mathrm{Im}\lambda| \leq \|v\|_2 + e^{\|q\|_1}.$$

Proof. Let $v \in H^2_c$ and recall that $Q = Q_1 \partial_x + Q_0$ with Q_1, Q_0 given by (1.7) and for any $F, G \in H^1([0,1], \mathbb{C}^4)$

$$\langle Q(v)F, G\rangle = [Q_1 F \cdot G]^1_0 + \langle F, Q(\overline{v})G\rangle$$

where $\langle\cdot,\cdot\rangle$ denotes the L^2 inner product, $\langle F,G\rangle = \int_0^1 F(x)\cdot\overline{G(x)}\,\mathrm{d}x$. On the domains (contained in $H^1([0,1],\mathbb{C}^4)$) of Q, corresponding to periodic, Dirichlet, or Neumann boundary conditions one has $[Q_1F\cdot G]_0^1 = 0$. In particular if λ is a periodic, Dirichlet, or Neumann eigenvalue and F a corresponding eigenfunction with $\langle F,F\rangle = 1$ one has

$$2\mathrm{iIm}\lambda = \lambda - \overline{\lambda} = \langle Q(v)F,F\rangle - \langle F,Q(v)F\rangle = \langle (Q(v)-Q(\overline{v}))F,F\rangle. \tag{3.21}$$

Note that $Q(v)-Q(\overline{v}) = 2\mathrm{iIm}Q_0(v)$ and hence by Cauchy-Schwarz and the normalization condition $\langle F,F\rangle = 1$

$$|\langle (Q(v)-Q(\overline{v}))F,F\rangle| \le \|2(\mathrm{Im}Q_0(v))F\|_{L^2}$$

where by (1.7) and (2.22)

$$\|2(\mathrm{Im}Q_0(v))F\|_{L^2} \le \frac{1}{2}(\|\mathrm{Im}Pp+q_x\|_{L^\infty} + \max_{\pm}\|\mathrm{Im}e^{\pm q/2}\|_{L^\infty}) \le \|v\|_2 + e^{\|q\|_1}$$

concluding the proof. □

Note that $\mu_m, \nu_m, \dot{\lambda}_m$ are close to $m\pi$ for $m\to\infty$. Our next aim is to obtain more precise asymptotics for these quantities. First we need to establish the following auxiliary result.

***Lemma* 3.14** *For any bi-infinite sequence of complex numbers $(\zeta_n)_n \subset \mathbb{C}^*$ satisfying $\zeta_n = n\pi + O(1)$ as $n\to\pm\infty$ one has*

$$\begin{aligned} \Delta|_{\lambda=\zeta_n} &= \cos(\zeta_n) + \ell_n^2, & \dot{\Delta}\Big|_{\lambda=\zeta_n} &= -\sin(\zeta_n) + \ell_n^2,\\ \delta|_{\lambda=\zeta_n} &= \ell_n^2, & \dot{\delta}\Big|_{\lambda=\zeta_n} &= \ell_n^2,\\ \chi_D|_{\lambda=\zeta_n} &= -\sin(\zeta_n) + \ell_n^2, & \dot{\chi}_D|_{\lambda=\zeta_n} &= -\cos(\zeta_n) + \ell_n^2. \end{aligned}$$

These estimates hold uniformly on subsets of sequences $(\zeta_n)_n$ where $(\omega(\zeta_n)-n\pi)_n$ is bounded. If in fact $\zeta_n = n\pi + \ell_n^2$, then $\sin(\zeta_n) = \ell_n^2$ and $\cos(\zeta_n) = (-1)^n + \ell_n^2$, yielding in particular the sharper asymptotics for $n\to\pm\infty$

$$\Delta|_{\lambda=\zeta_n} = (-1)^n + \ell_n^2, \qquad \dot{\Delta}\Big|_{\lambda=\zeta_n} = \ell_n^2.$$

Proof. The stated asymptotics follow from Theorem 2.12(iii) and (iv). □

***Lemma* 3.15** *For any $v\in H_c^1$, the roots of $\dot{\Delta}$ in the half plane $\mathbb{C}^+$ (defined in (3.9)) have the following asymptotics as $n\to\infty$*

$$\dot{\lambda}_n = n\pi + \ell_n^2, \quad \frac{1}{16\dot{\lambda}_{-n}} = n\pi + \ell_n^2.$$

These estimates hold locally uniformly on H_c^1.

Proof. Since by Lemma 3.12, $\dot{\lambda}_n = n\pi + O(1)$, it follows from Lemma 3.14, that

$$0 = \dot{\Delta}(\dot{\lambda}_n) = -\sin(\dot{\lambda}_n) + \ell_n^2, \tag{3.22}$$

or $\sin(\dot{\lambda}_n) = \ell_n^2$. Since by Lemma 3.12, $|\dot{\lambda}_n - n\pi| < \pi/3$ for $|n|$ large enough, one has

$$|\dot{\lambda}_n - n\pi|\cos(\pi/3) \le |\dot{\lambda}_n - n\pi|\left|\int_0^1 \cos((\dot{\lambda}_n - n\pi)s + n\pi)\,\mathrm{d}s\right| = \left|\sin(\dot{\lambda}_n) - \sin(n\pi)\right| = \ell_n^2$$

proving the first claimed asymptotics. They in turn yield the second ones by Lemma 2.14(ii) (reciprocity in λ). □

By the same arguments one can prove that similar results hold for the Dirichlet eigenvalues.

***Lemma* 3.16** *For any $v\in H_c^1$, the Dirichlet eigenvalues of $Q(v)$ in $\mathbb{C}^+$ have the following asymptotics as $n\to\infty$*

$$\mu_n = n\pi + \ell_n^2, \quad \frac{1}{16\mu_{-n}} = n\pi + \ell_n^2.$$

These estimates hold locally uniformly in H_c^1.

We will now use Lemma 3.15 and Lemma 3.16 to prove the following result for the periodic eigenvalues of $Q(v)$.

Lemma 3.17 *For any $v \in H^1_c$, the periodic eigenvalues of $Q(v)$ in $\mathbb{C}^+$ have the following asymptotics as $n \to \infty$*

$$\lambda_n^\pm = n\pi + \ell_n^2 \quad \textit{and} \quad \frac{1}{16\lambda_{-n}^\pm} = n\pi + \ell_n^2.$$

These estimates hold locally uniformly on H^1_c.

Proof. Let $v \in H^1_c$ be given. Since by Lemma 3.16 $\mu_n = n\pi + \ell_n^2$, Lemma 3.14 yields $\delta(\mu_n) = \ell_n^2$. Hence by Lemma 3.8 one has

$$\Delta^2(\mu_n) - 1 = \delta^2(\mu_n) = \ell_n^1.$$

On the other hand, Lemma 3.14 also yields that $\Delta(\mu_n) = (-1)^n + \ell_n^2$. Writing $\Delta^2(\mu_n) - 1 = (\Delta(\mu_n) - 1)(\Delta(\mu_n) + 1)$ the two latter estimates together imply that

$$\Delta(\mu_n) = (-1)^n + \ell_n^1. \tag{3.23}$$

A similar estimate holds for $\Delta(\dot\lambda_n)$. Indeed, since $\dot\lambda_n - \mu_n = \ell_n^2$ by Lemma 3.15 and Lemma 3.16,

$$\Delta(\dot\lambda_n) - \Delta(\mu_n) = (\dot\lambda_n - \mu_n) \int_0^1 \dot\Delta(t\dot\lambda_n + (1-t)\mu_n)\,\mathrm{d}t,$$

and $\dot\Delta(t\dot\lambda_n + (1-t)\mu_n) = \ell_n^2$ uniformly in $0 \le t \le 1$ by Lemma 3.14 , it follows that $\Delta(\dot\lambda_n) - \Delta(\mu_n) = \ell_n^1$ which together with (3.23) yields

$$\Delta(\dot\lambda_n) = (-1)^n + \ell_n^1.$$

The latter estimates can be applied as follows. Since $\dot\Delta(\dot\lambda_n) = 0$ one has $\Big[(1-t)\dot\Delta(t\lambda_n^\pm + (1-t)\dot\lambda_n)\Big]_0^1 = 0$ and therefore integrating by parts,

$$\begin{aligned}\Delta(\lambda_n^\pm) - \Delta(\dot\lambda_n) =& (\lambda_n^\pm - \dot\lambda_n) \int_0^1 \dot\Delta(t\lambda_n^\pm + (1-t)\dot\lambda_n)\,\mathrm{d}t \\ =& (\lambda_n^\pm - \dot\lambda_n)^2 \int_0^1 (1-t)\ddot\Delta(t\lambda_n^\pm + (1-t)\dot\lambda_n)\,\mathrm{d}t.\end{aligned}$$

Hence

$$(\lambda_n^\pm - \dot\lambda_n)^2 \int_0^1 (1-t)\ddot\Delta(t\lambda_n^\pm + (1-t)\dot\lambda_n)\,\mathrm{d}t = \ell_n^1. \tag{3.24}$$

Since Δ is analytic in λ and $\dot\Delta(\zeta_n) = -\sin(\zeta_n) + \ell_n^2$ by Lemma 3.14, Cauchy's estimate yields

$$\ddot\Delta(t\lambda_n^\pm + (1-t)\dot\lambda_n) = -\cos(t\lambda_n^\pm + (1-t)\dot\lambda_n) + \ell_n^2$$

uniformly in $0 \le t \le 1$. For n sufficiently large, $t\lambda_n^\pm + (1-t)\dot\lambda_n$ is in D_n and hence $\int_0^1 (1-t)\ddot\Delta(t\lambda_n^\pm + (1-t)\dot\lambda_n)\,\mathrm{d}t$ is uniformly bounded away from zero for such n. So (3.24) yields

$$\lambda_n^\pm - \dot\lambda_n = \ell_n^2.$$

Since by Lemma 3.15, $\dot\lambda_n = n\pi + \ell_n^2$ the first claimed asymptotics follow. Those then yield the second ones by Lemma 2.14(ii) (reciprocity in λ). ☐

4 Spectral gaps

For any potential v in H^1_c we denote by

$$G_n \equiv G_n(v) := [\lambda_n^-, \lambda_n^+] := \{\ (1-t)\lambda_n^- + t\lambda_n^+ \ :\ 0 \le t \le 1\ \}, \quad n \in \mathbb{Z} \tag{4.1}$$

the straight line segment in the complex plane between the periodic eigenvalues λ_n^- and λ_n^+ of $Q_1\partial_x + Q_0$. By a slight abuse of terminology we refer to G_n as the n-th closed spectral gap, although for $(q,p) \notin H_r^1$ it lacks a spectral interpretation. Furthermore for any $(q,p) \in H_c^1$ and $n \in \mathbb{Z}$ we introduce the gap length γ_n

$$\gamma_n(v) := \lambda_n^+(v) - \lambda_n^-(v), \quad n \in \mathbb{Z}. \tag{4.2}$$

Note that in general, $\gamma_n(v)$ is a complex number but in case $v \in H_r^1$, it is real and equals the length of the gap $G_n(v)$.

One of the main results of this chapter states that for $v \in H_c^{s+1}$, $s \geq 0$, the sequence $(\gamma_n(v))_{n\geq 0}$ is in $\ell^{2,s}(\mathbb{Z}_{\geq 0}, \mathbb{C})$ (cf Theorem 4.10). A potential $v \in H_c^1$ is said to be a right [left] sided N-gap potential with $N \in \mathbb{Z}_{\geq 0}$ if

$$\gamma_n(v) = 0 \quad \forall\, n > N \quad [\gamma_{-n}(v) = 0 \quad \forall\, n > N]. \tag{4.3}$$

It is said to be a right [left] sided finite gap potential if it is a right [left] sided N-gap potential for some $N \in \mathbb{Z}$. Another result we prove in this chapter is that the set of right [left] sided finite gap potentials in H_c^s is dense in H_c^s for any $s \in \mathbb{R}_{\geq 1}$.

The proofs of the results of this chapter are based on a *Lyapunov-Schmidt decomposition* developed in previous work for the Hill and Zakharov-Shabat operators - see [3], [7],[9], [19] and references therein.

4.1 Lyapunov-Schmidt decomposition

For $d = 1, 2, 4$ and $s \in \mathbb{R}_{\geq 0}$ denote by $H^s(\mathbb{T}_2, \mathbb{C}^d)$ the Sobolev space of order s of two periodic functions with values in $\mathbb{C}^d$,

$$H^s(\mathbb{T}_2, \mathbb{C}^d) := \{\, u = \sum_{n\in\mathbb{Z}} u_n e_n \,:\, u_n \in \mathbb{C}^d \text{ and } \|u\|_s < \infty \,\}, \quad \|u\|_s := \Big(\sum_{n\in\mathbb{Z}} \langle n\rangle^{2s} |u_n|^2\Big)^{1/2}$$

where $\mathbb{T}_2 = \mathbb{R}/\mathbb{Z}$, $e_n(x) = e^{\mathrm{i}n\pi x}$, and $|a| = \Big(\sum_{j=1}^d a_j^2\Big)^{1/2}$ for any $a = (a_1, \dots, a_d) \in \mathbb{C}^d$. We recall that the weights $\langle n\rangle^s := (1 + \pi^2 n^2)^{s/2}$ are submultiplicative for any $s \geq 0$, i.e. $\langle n+m\rangle^s \leq \langle n\rangle^s \langle m\rangle^s$. The L^2-inner product is defined for $f, g \in H^0(\mathbb{T}_2, \mathbb{C}^d) \equiv L^2(\mathbb{T}_2, \mathbb{C}^d)$ by

$$\langle f, g\rangle_c = \frac{1}{2}\int_0^2 f\overline{g}\,\mathrm{d}x = \frac{1}{2}\int_0^2 \sum_{j=1}^d f^{(j)}(x)\overline{g^{(j)}(x)}\,\mathrm{d}x \tag{4.4}$$

where $f^{(j)}$, $1 \leq j \leq d$, denote the components of f.

For a scalar valued function $u \in H^s(\mathbb{T}_2, \mathbb{C})$ and a vector valued function $v \in H^s(\mathbb{T}_2, \mathbb{C}^d)$ with $v = \sum_{n\in\mathbb{Z}} v_n e_n$ and $v_n \in \mathbb{C}^d$ one has, using Young's inequality and $\langle n\rangle^s \leq \langle k\rangle^s \langle n-k\rangle^s$,

$$\|uv\|_s \leq \|u\|_s \sum_{n\in\mathbb{Z}} \langle n\rangle^s |v_n|. \tag{4.5}$$

Hence by the Cauchy-Schwarz inequality and $\sum_{n\in\mathbb{Z}} \frac{1}{\langle n\rangle^2} \leq 1 + \frac{2}{\pi^2}\sum_{n=1}^\infty \frac{1}{n^2}$, where

$$\sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{6}, \tag{4.6}$$

one has

$$\|uv\|_s \leq 2\|u\|_s \|v\|_{s+1}. \tag{4.7}$$

Note that the estimate (4.7) can be easily improved, but for our purpose it suffices. Recall that by (2.5)-(2.6) $\mathcal{Q}(q,p) = \mathcal{Q}_1\partial_x + \mathcal{Q}_0(q,p)$ with

$$\mathcal{Q}_1 = \begin{pmatrix} R & \\ & \end{pmatrix}, \quad \mathcal{Q}_0(q,p) = \begin{pmatrix} \mathcal{A}(q,p) & \mathcal{B}(q,p) \\ \mathcal{B}(q,p) & \end{pmatrix},$$

and

$$\mathcal{A}(q,p) = \frac{1}{4}\varphi J,\ \varphi := -\mathrm{i}(Pp + q_x), \quad \mathcal{B}(q,p) = \frac{1}{4}\begin{pmatrix} \cosh(q/2) & -\sinh(q/2) \\ -\sinh(q/2) & \cosh(q/2) \end{pmatrix}.$$

It turns out to be useful to introduce the linear isomorphism

$$H^{s+1}_{\mathbb{C}} \times H^{s+1}_{\mathbb{C}} \to H^{s+1}_{\mathbb{C}} \times H^{s}_{\mathbb{C}}, (q,p) \mapsto (q,\varphi)$$

and use (q,φ) instead of (q,p) as phase space variables. Furthermore introduce

$$\tilde{H}^{s+1}_{c} := H^{s+1}_{\mathbb{C}} \times H^{s}_{\mathbb{C}} \tag{4.8}$$

and by a slight abuse of notation we write $\mathcal{Q} \equiv \mathcal{Q}(q,\varphi)$ and $\mathcal{Q}_0 \equiv \mathcal{Q}_0(q,\varphi)$ for the operators $\mathcal{Q}(q,p)$ and $\mathcal{Q}_0(q,p)$ respectively. We rewrite equation (2.10) in the form

$$\mathcal{Q}(\lambda)F = \mathcal{Q}_0 F, \quad \mathcal{Q}(\lambda) := -\mathcal{Q}_1\partial_x + \lambda I \tag{4.9}$$

and introduce the L^2-orthogonal basis of $L^2(\mathbb{T}_2, \mathbb{C}^4)$,

$$e^{(j)}_n(x) = e_n(x)a^{(j)},\ 1 \le j \le 4,\ n \in \mathbb{Z}, \quad e_n(x) = e^{in\pi x}$$

where $a^{(1)}, a^{(2)}, a^{(3)}, a^{(4)}$ denote the standard basis in $\mathbb{C}^4$, $a^{(1)} = (1,0,0,0)$, $a^{(2)} = (0,1,0,0)$, $a^{(3)} = (0,0,1,0)$, $a^{(4)} = (0,0,0,1)$. Note that

$$\mathcal{Q}(\lambda)e^{(1)}_n = (\lambda + n\pi)e^{(1)}_n \qquad \mathcal{Q}(\lambda)e^{(2)}_n = (\lambda - n\pi)e^{(2)}_n \tag{4.10}$$

$$\mathcal{Q}(\lambda)e^{(3)}_n = \lambda e^{(3)}_n \qquad \mathcal{Q}(\lambda)e^{(4)}_n = \lambda e^{(4)}_n \tag{4.11}$$

suggesting to decompose $H^s(\mathbb{T}_2, \mathbb{C}^4)$ with $s \ge 0$ for any given $n \in \mathbb{Z}$ as $H^s(\mathbb{T}_2, \mathbb{C}^4) = \mathcal{P}_n \oplus \mathcal{K}_n$ where

$$\begin{aligned}\mathcal{P}_n :=& \{\ f^{(1)}_{-n}e^{(1)}_{-n} + f^{(2)}_n e^{(2)}_n \ :\ f^{(1)}_{-n}, f^{(2)}_n \in \mathbb{C}\ \},\\ \mathcal{K}_n :=& \{ \sum_{k\in\mathbb{Z}, 1\le j\le 4} f^{(j)}_k e^{(j)}_k \in H^s(\mathbb{T}_2, \mathbb{C}^4) \ :\ f^{(j)}_k \in \mathbb{C}, f^{(1)}_{-n} = 0,\ f^{(2)}_n = 0\ \}.\end{aligned}$$

Denote the L^2-orthogonal projections onto $\mathcal{P}_n$ and $\mathcal{K}_n$ by P_n and K_n, respectively. The subspaces $\mathcal{P}_n$ and $\mathcal{K}_n$ are invariant under $\mathcal{Q}(\lambda)$. Furthermore, introduce for any $n \in \mathbb{Z}$ the complex strip

$$\Pi_n = \{\ \lambda \in \mathbb{C} \ :\ |\mathrm{Re}\lambda - n\pi| \le \pi/2\ \}. \tag{4.12}$$

Note that these strips cover $\mathbb{C}$ and that for any $n \ne 0$ the restriction of $\mathcal{Q}(\lambda)$ to $\mathcal{K}_n$, again denoted by $\mathcal{Q}(\lambda)$, is invertible for any $\lambda \in \Pi_n$. Writing $F = u + v$ with $u := P_n F$ and $v := K_n F$, equation (4.9) decomposes into the following system of equations

$$\mathcal{Q}(\lambda)u = P_n\mathcal{Q}_0(u + v) \tag{4.13}$$

$$\mathcal{Q}(\lambda)v = K_n\mathcal{Q}_0(u + v), \tag{4.14}$$

called P- and K-equation, respectively. (Note that in this section, u and v have a different meaning than elsewhere.) Given any $n \ne 0$ we first solve the K-equation for any given $u \in \mathcal{P}_n$ and then substitute the solution into the P-equation, leading to a 2×2 system of linear equations with a 2×2 coefficient matrix S_n, which is singular precisely when λ is a periodic eigenvalue of $\mathcal{Q}$. The proof of Theorem 4.15 will follow by analyzing the coefficients of S_n.

Actually, to solve (4.13), it suffices to determine $\mathcal{Q}_0 v$. Hence in a first step, we derive from (4.14) an equation for $\mathcal{Q}_0 v$ instead of v. Once u and $\mathcal{Q}_0 v$ are found v can be easily determined from (4.14), $v = \mathcal{Q}(\lambda)^{-1}K_n(\mathcal{Q}_0 u + \mathcal{Q}_0 v)$. We begin by deriving from (4.14) an equation for $\mathcal{Q}_0 v$. Given any $n \ne 0$ and $\lambda \in \Pi_n$, apply the operator $\mathcal{Q}_0\mathcal{Q}(\lambda)^{-1}$ to (4.14) to obtain,

$$\mathcal{Q}_0 v = \mathcal{Q}_0\mathcal{Q}(\lambda)^{-1}K_n\mathcal{Q}_0(u + v) \tag{4.15}$$

which leads to the following equation for $\tilde{v} = \mathcal{Q}_0 v \in L^2(\mathbb{T}_2, \mathbb{C}^4)$,

$$(Id - T_n)\tilde{v} = T_n\mathcal{Q}_0 u, \quad T_n := \mathcal{Q}_0\mathcal{Q}(\lambda)^{-1}K_n : L^2(\mathbb{T}_2, \mathbb{C}^4) \to L^2(\mathbb{T}_2, \mathbb{C}^4).$$

We then prove that for $|n|$ sufficiently large, T^4_n is a contraction implying that for such n, $Id - T^4_n$ is invertible. The invertibility of the operator $Id - T_n$ then follows from the identity

$$(Id - T_n)^{-1} = (Id + T_n)(Id + T^2_n)(Id - T^4_n)^{-1}.$$

First we need to introduce some more notation. For $d = 1, 2, 4$ and $s \ge 0$ we consider on $H^s(\mathbb{T}_2, \mathbb{C}^d)$ the shifted norms

$$\|u\|_{s;n} := \|ue_n\|_s = \big(\sum_{k\in\mathbb{Z}} \langle k + n\rangle^{2s}|u_n|^2\big)^{1/2}$$

Note that the estimate (4.7) continues to hold for these norms. More precisely, the following holds:

***Lemma* 4.1** *Let $s \in \mathbb{R}_{\geq 0}$ and $n \in \mathbb{Z}$. Then the following holds:*

(i) for any $u \in H^s(\mathbb{T}_2, \mathbb{C})$ and $v \in H^{s+1}(\mathbb{T}_2, \mathbb{C}^d)$

$$\|uv\|_{s;n} \leq 2\|u\|_{s;n}\|v\|_{s+1}, \quad \|uv\|_{s;n} \leq 2\|u\|_s\|v\|_{s+1;n}$$

(ii) for any $u \in H^{s+1}(\mathbb{T}_2, \mathbb{C})$ and $v \in H^s(\mathbb{T}_2, \mathbb{C}^d)$

$$\|uv\|_{s;n} \leq 2\|u\|_{s+1;n}\|v\|_s, \quad \|uv\|_{s;n} \leq 2\|u\|_{s+1}\|v\|_{s;n}.$$

Proof. One computes

$$\|uv\|_{s;n} = \|uve_n\|_s \leq 2\|ue_n\|_s\|v\|_{s+1} = 2\|u\|_{s;n}\|v\|_{s+1}.$$

The other inequalities are obtained in a similar fashion. □

***Lemma* 4.2** *For any $(q, \varphi) \in \tilde{H}_c^{s+1}$ with $s \geq 0$, $l \in \mathbb{Z}$, and $\lambda \in \Pi_n$ with $n \neq 0$, the following holds:*

(i) Decomposing

$$T_n = \mathcal{Q}_0\mathcal{Q}(\lambda)^{-1}K_n : (H^s(\mathbb{T}_2, \mathbb{C}^4), \|\cdot\|_{s;l}) \to (H^s(\mathbb{T}_2, \mathbb{C}^4), \|\cdot\|_{s;l})$$

according to

$$\mathcal{Q}_0 = \begin{pmatrix} \mathcal{A} & \\ & \end{pmatrix} + \begin{pmatrix} & \\ \mathcal{B} & \end{pmatrix} + \begin{pmatrix} & \mathcal{B} \\ & \end{pmatrix},$$

the resulting operators satisfy

$$\left\| \begin{pmatrix} \mathcal{A} & \\ & \end{pmatrix} \mathcal{Q}(\lambda)^{-1}K_n \right\|_{s;l} \leq \|\varphi\|_s, \quad \left\| \begin{pmatrix} & \\ \mathcal{B} & \end{pmatrix} \mathcal{Q}(\lambda)^{-1}K_n \right\|_{s;l} \leq \|\sinh(q/2)\|_s + \|\cosh(q/2)\|_s, \tag{4.16}$$

$$\left\| \begin{pmatrix} & \mathcal{B} \\ & \end{pmatrix} \mathcal{Q}(\lambda)^{-1}K_n \right\|_{s;l} \leq \frac{\|\sinh(q/2)\|_{s+1} + \|\cosh(q/2)\|_{s+1}}{|n|} \tag{4.17}$$

(ii) T_n is bounded with

$$\|T_n\|_{s;l} \leq R_s, \quad R_s \equiv R_s(q, \varphi) := \|\varphi\|_s + \|\sinh(q/2)\|_{s+1} + \|\cosh(q/2)\|_{s+1}. \tag{4.18}$$

Proof. Let $n \neq 0$ clearly (4.18) follows from (4.16)-(4.17). The latter two estimates are proved separately. To prove (4.16) note that for $m \neq n$

$$\min_{\lambda \in \Pi_n} |\lambda - m\pi| \geq |n - m| \geq 1, \tag{4.19}$$

implying that for any $\lambda \in \Pi_n$, $n \neq 0$, the restriction of $\mathcal{Q}(\lambda)$ to the invariant subspace $\mathcal{K}_n$ is invertible and that its inverse is uniformly bounded for $\lambda \in \Pi_n$. For any $F = \sum_{m \in \mathbb{Z}, j=1,2,3,4} f_m^{(j)} e_m^{(j)}$ with $f_m^{(j)} \in \mathbb{C}$, and $\lambda \in \Pi_n$

$$\mathcal{Q}(\lambda)^{-1}K_nF = \sum_{m \neq n} \frac{1}{\lambda - m\pi} \left(f_{-m}^{(1)}e_{-m}^{(1)} + f_m^{(2)}e_m^{(2)}\right) + \frac{1}{\lambda} \sum_{m \in \mathbb{Z}} \left(f_m^{(3)}e_m^{(3)} + f_m^{(4)}e_m^{(4)}\right)$$

is well defined. Taking into account Lemma 4.1 and the definition (2.7) of $\mathcal{A}$ one has

$$\begin{aligned}\left\| \begin{pmatrix} \mathcal{A} & \\ & \end{pmatrix} \mathcal{Q}(\lambda)^{-1}\mathcal{K}_nF \right\|_{s;l} &= \left\| \begin{pmatrix} \mathcal{A} & \\ & \end{pmatrix} \sum_{m \neq n} \frac{1}{\lambda - m\pi} \left(f_{-m}^{(1)}e_{-m}^{(1)} + f_m^{(2)}e_m^{(2)}\right) \right\|_{s;l} \\ &\leq \frac{1}{4}\|\varphi\|_s \sum_{m \neq n} \frac{|f_{-m+l}^{(1)}|\langle -m+l \rangle^s + |f_{m+l}^{(2)}|\langle m+l \rangle^s}{|n-m|}.\end{aligned}$$

For $1 \leq j \leq 4$, let

$$f^{(j)} := \sum_{m \in \mathbb{Z}} f_m^{(j)} e_m^{(j)}.$$

By the Cauchy-Schwarz inequality and (4.6) one then concludes

$$\sum_{m\neq n} \frac{|f^{(1)}_{-m+l}|\langle -m+l\rangle^s + |f^{(2)}_{m+l}|\langle m+l\rangle^s}{|n-m|} \leq \|f^{(1)}+f^{(2)}\|_{s;l}\sqrt{\sum_{m\neq n}\frac{1}{|m-n|^2}} \leq \frac{\pi}{\sqrt{3}}\|F\|_{s;l}.$$

where we used that $\|f^{(1)}+f^{(2)}\|^2_{s;l} = \|f^{(1)}\|^2_{s;l} + \|f^{(2)}\|^2_{s;l} \leq \|F\|^2_{s;l}$. A similar estimate holds for $\left\|\begin{pmatrix} & \\ \mathcal{B} & \end{pmatrix}\mathcal{Q}(\lambda)^{-1}\mathcal{K}_n F\right\|_{s;l}$ where $\frac{1}{4}\|\varphi\|_s$ is replaced by $\frac{1}{4}\|\sinh(q/2)\|_s + \frac{1}{4}\|\cosh(q/2)\|_s$. Altogether this yields the two estimates of (4.16). On the other hand

$$\begin{pmatrix} & \mathcal{B} \\ & \end{pmatrix}\mathcal{Q}(\lambda)^{-1}\mathcal{K}_n F = \begin{pmatrix} & \mathcal{B} \\ & \end{pmatrix}\frac{1}{\lambda}\sum_{m\in\mathbb{Z}}\left(f^{(3)}_m e^{(3)}_m + f^{(4)}_m e^{(4)}_m\right).$$

Since by (4.19) $\frac{1}{|\lambda|} \leq \frac{1}{|n|}$ for any $\lambda \in \Pi_n$, estimate (4.17) then follows from Lemma 4.1. □

Lemma 4.3 *Let $(q,\varphi) \in \tilde{H}^{s+1}_c$ with $s \geq 0$ and $\lambda \in \Pi_n$ with $n \in \mathbb{Z}\setminus\{0\}$. Then the following holds:*

(i) For any $F = \sum_{j=1}^4 f^{(j)} \in H^s(\mathbb{T}_2,\mathbb{C}^4)$ the following estimates hold:

$$\left\|T_n\begin{pmatrix}\mathcal{A} & \\ & \end{pmatrix}\mathcal{Q}(\lambda)^{-1}K_n f^{(1)}\right\|_{s;-n} \leq \frac{1}{2|n|}R_s(q,\varphi)\|\varphi\|_s\|f^{(1)}\|_{s;-n}, \tag{4.20}$$

and

$$\left\|T_n\begin{pmatrix}\mathcal{A} & \\ & \end{pmatrix}\mathcal{Q}(\lambda)^{-1}K_n f^{(2)}\right\|_{s;n} \leq \frac{1}{2|n|}R_s(q,\varphi)\|\varphi\|_s\|f^{(2)}\|_{s;n}, \tag{4.21}$$

while for $j = 3,4$

$$T_n\begin{pmatrix}\mathcal{A} & \\ & \end{pmatrix}\mathcal{Q}(\lambda)^{-1}K_n f^{(j)} = 0.$$

(ii) Furthermore

$$\left\|\left[\begin{pmatrix}\mathcal{A} & \\ & \end{pmatrix}\mathcal{Q}(\lambda)^{-1}K_n\right]^3\right\|_{s;\pm n} \leq \frac{1}{|n|}\|\varphi\|^3_s. \tag{4.22}$$

(iii) For any $F = \sum_{j=1}^4 f^{(j)} \in H^s(\mathbb{T}_2,\mathbb{C}^4)$ the following estimates hold:

$$\left\|T_n\begin{pmatrix}\mathcal{A} & \\ & \end{pmatrix}\mathcal{Q}(\lambda)^{-1}K_n\right\|_{s;\pm n} \leq R_s(q,\varphi)\big(\frac{2}{\sqrt{|n|}}\|\varphi\|_s + R_{s;|n|}(\varphi)\big), \tag{4.23}$$

where for any $g = \sum_{k\in\mathbb{Z}} g_k e_k \in H^s(\mathbb{T}_2,\mathbb{C})$,

$$R_{s;|n|}(g) := \sqrt{\sum_{|k|\geq|n|}\langle k\rangle^{2s}|g_k|^2}. \tag{4.24}$$

Proof. (i) Writing $\varphi = \sum_{m\in\mathbb{Z}}\varphi_m e_m$, a straightforward computation yields for $F = \sum_{j=1}^4 f^{(j)} \in H^s(\mathbb{T},\mathbb{C}^4)$

$$g^{(1)} := \mathcal{Q}(\lambda)^{-1}K_n\begin{pmatrix}\mathcal{A} & \\ & \end{pmatrix}\mathcal{Q}(\lambda)^{-1}K_n f^{(2)} = \frac{1}{4}\sum_{k\neq -n}\sum_{m\neq n}\frac{f^{(2)}_m\varphi_{k-m}}{(\lambda-m\pi)(\lambda+k\pi)}e^{(1)}_k \tag{4.25}$$

and similarly

$$g^{(2)} := \mathcal{Q}(\lambda)^{-1}K_n\begin{pmatrix}\mathcal{A} & \\ & \end{pmatrix}\mathcal{Q}(\lambda)^{-1}K_n f^{(1)} = \frac{1}{4}\sum_{k\neq n}\sum_{m\neq -n}\frac{-f^{(1)}_m\varphi_{k-m}}{(\lambda+m\pi)(\lambda-k\pi)}e^{(2)}_k. \tag{4.26}$$

while for $j = 3,4$

$$\begin{pmatrix}\mathcal{A} & \\ & \end{pmatrix}\mathcal{Q}(\lambda)^{-1}K_n f^{(j)} = 0. \tag{4.27}$$

Note that the coefficients of $g^{(1)} = \sum_{k\neq -n} g_k^{(1)} e_k^{(1)}$ and $g^{(2)} = \sum_{k\neq n} g_k^{(2)} e_k^{(2)}$ are given by

$$g_k^{(1)} = \frac{1}{4}\sum_{m\neq n} \frac{f_m^{(2)}\varphi_{k-m}}{(\lambda - m\pi)(\lambda + k\pi)}, \quad g_k^{(2)} = -\frac{1}{4}\sum_{m\neq -n} \frac{f_m^{(1)}\varphi_{k-m}}{(\lambda + m\pi)(\lambda - k\pi)}$$

By Lemma 4.1 one has for $j = 1, 2$ and $i = 1, 2$ such that $\{1, 2\} = \{i, j\}$

$$\begin{aligned}\left\| T_n \begin{pmatrix} \mathcal{A} & \\ & \end{pmatrix} \mathcal{Q}(\lambda)^{-1} K_n f^{(j)} \right\|_{s;l} =& \left\| \mathcal{Q}_0 g^{(i)} \right\|_{s;l} \\ \leq& \frac{1}{4}\Big(\|\varphi\|_s + \|\sinh(q/2)\|_s + \|\cosh(q/2)\|_s \Big) \left\| g^{(i)} e_l \right\|_{W^{s,1}}\end{aligned}$$

where

$$\|g^{(i)}\|_{W^{s,1}} := \sum_{m\in\mathbb{Z}} \langle m\rangle^s |g_m^{(i)}|. \tag{4.28}$$

The bounds (4.20) and (4.21) then follow from corresponding bounds of $\|g^{(1)}e_n\|_{W^{s,1}}$ and $\|g^{(2)}e_{-n}\|_{W^{s,1}}$. For $g^{(1)}e_n$ one has

$$\|g^{(1)}e_n\|_{W^{s,1}} = \left\| \frac{1}{4}\sum_{k\neq -n}\sum_{m\neq n} \frac{f_m^{(2)}\varphi_{k-m}}{(\lambda - m\pi)(\lambda + k\pi)} e_k^{(1)} e_n \right\|_{W^{s,1}} \leq \frac{1}{4}\sum_{k\neq -n}\sum_{m\neq n} \frac{\langle k+n\rangle^s}{|n-m||n+k|} |f_m^{(2)}||\varphi_{k-m}|. \tag{4.29}$$

Since for $k \neq -n$, $\frac{\langle k+n\rangle^s}{|k+n|} \leq 4\langle k+n\rangle^{s-1}$ and $\langle k+n\rangle^{s-1} \leq \langle m+n\rangle^{s-1}\langle k-m\rangle^{s-1}$ one obtains by Young's inequality

$$\frac{1}{4}\sum_{k\neq -n}\sum_{m\neq n} \frac{\langle k+n\rangle^s}{|n-m||n+k|} |f_m^{(2)}||\varphi_{k-m}| \leq \sum_{m\neq n} \frac{|f_m^{(2)}|\langle m+n\rangle^{s-1}}{|n-m|} \sum_{k\in\mathbb{Z}} \langle k\rangle^{s-1}|\varphi_k|.$$

By the Cauchy-Schwarz inequality

$$\sum_{m\neq n} \frac{|f_m^{(2)}|\langle m+n\rangle^{s-1}}{|n-m|} = \sum_{m\neq n} \frac{1}{\langle m+n\rangle|n-m|} |f_m^{(2)}|\langle m+n\rangle^s \leq \|f^{(2)}\|_{s;n} \Big(\sum_{m\neq n} \frac{1}{\langle m+n\rangle^2|n-m|^2} \Big)^{1/2}. \tag{4.30}$$

For $n > 0$ one has

$$\sum_{m\neq n} \frac{1}{|n-m|^2\langle m+n\rangle^2} \leq \frac{1}{\pi^2 n^2} + \frac{1}{\pi^2 n^2}\sum_{m<0}\frac{1}{m^2} + \sum_{m>0, m\neq n} \frac{1}{|n-m|^2\pi^2 n^2} \leq \frac{1}{n^2}\Big(\frac{1}{\pi^2} + \frac{3}{\pi^2}\sum_{m>0}\frac{1}{m^2}\Big). \tag{4.31}$$

Since a similar statement holds for $n < 0$ and

$$\sum_{k\in\mathbb{Z}} \langle k\rangle^{s-1}|\varphi_k| \leq \sqrt{\sum_{k\in\mathbb{Z}} \frac{1}{\langle k\rangle^2}} \sqrt{\sum_{k\in\mathbb{Z}} \langle k\rangle^{2s}|\varphi_k|^2} \leq \|\varphi\|_s \sqrt{1 + \frac{1}{3}}$$

one has by (4.6)

$$\|g^{(1)}e_n\|_{W^{s,1}} \leq \|\varphi\|_s \frac{2}{|n|} \|f^{(2)}\|_{s;n}. \tag{4.32}$$

A similar computation yields

$$\|g^{(2)}e_{-n}\|_{W^{s,1}} \leq \|\varphi\|_s \frac{2}{|n|} \|f^{(1)}\|_{s;-n}. \tag{4.33}$$

This proves (4.21) and (4.20).

(ii) To prove (4.22), note that by (4.30) the definition (4.25) of $g^{(1)}$ amd the estimate (4.5)

$$\left\| \left[\begin{pmatrix} \mathcal{A} & \\ & \end{pmatrix} \mathcal{Q}(\lambda)^{-1} K_n \right]^2 f^{(2)} \right\|_{s;\pm n} = \left\| \begin{pmatrix} \mathcal{A} & \\ & \end{pmatrix} g^{(1)} \right\|_{s;\pm n} \leq \frac{1}{4}\|\varphi\|_s \|g^{(1)}e_{\pm n}\|_{W^{s,1}}$$

hence by (4.32)

$$\left\| \left[\begin{pmatrix} \mathcal{A} & \\ & \end{pmatrix} \mathcal{Q}(\lambda)^{-1} K_n \right]^2 f^{(2)} \right\|_{s;\pm n} \leq \frac{1}{2|n|} \|\varphi\|_s^2 \|f^{(2)}\|_{s;\pm n}. \tag{4.34}$$

It then follows by (4.16) that

$$\left\| \left[\begin{pmatrix} \mathcal{A} & \\ & \end{pmatrix} \mathcal{Q}(\lambda)^{-1} K_n \right]^3 f^{(2)} \right\|_{s;\pm n} \leq \|\varphi\|_s \left\| \left[\begin{pmatrix} \mathcal{A} & \\ & \end{pmatrix} \mathcal{Q}(\lambda)^{-1} K_n \right]^2 f^{(2)} \right\|_{s;\pm n} \leq \frac{1}{2|n|} \|\varphi\|_s^3 \|f^{(2)}\|_{s;\pm n}.$$

By the definition of $\mathcal{A} = \frac{1}{4}\varphi J$, all components of

$$\begin{pmatrix} \mathcal{A} & \\ & \end{pmatrix} \mathcal{Q}(\lambda)^{-1} K_n f^{(1)}$$

vanish except the second one, hence (4.34) implies

$$\left\| \left[\begin{pmatrix} \mathcal{A} & \\ & \end{pmatrix} \mathcal{Q}(\lambda)^{-1} K_n \right]^3 f^{(1)} \right\|_{s;\pm n} \leq \frac{1}{2|n|} \|\varphi\|_s^2 \left\| \begin{pmatrix} \mathcal{A} & \\ & \end{pmatrix} \mathcal{Q}(\lambda)^{-1} K_n f^{(1)} \right\|_{s;\pm n}.$$

By (4.16) it then follows that

$$\left\| \left[\begin{pmatrix} \mathcal{A} & \\ & \end{pmatrix} \mathcal{Q}(\lambda)^{-1} K_n \right]^3 f^{(1)} \right\|_{s;\pm n} \leq \frac{1}{2|n|} \|\varphi\|_s^3 \|f^{(1)}\|_{s;\pm n}.$$

In view of (4.27) the claimed estimate (4.22) then follows.

(iii) In view of item (i) and the definitions (4.25) and (4.26) of $g^{(1)}$ and $g^{(2)}$, it remains to bound $\|g^{(2)} e_n\|_{W^{s,1}}$ and $\|g^{(1)} e_{-n}\|_{W^{s,1}}$. One computes

$$\|g^{(2)} e_n\|_{W^{s,1}} = \left\| \frac{1}{4} \sum_{k \neq n} \sum_{m \neq -n} \frac{-f_m^{(1)} \varphi_{k-m}}{(\lambda + m\pi)(\lambda - k\pi)} e_k^{(2)} e_n \right\|_{W^{s,1}} \leq \frac{1}{4} \sum_{k \neq n} \sum_{m \neq -n} \frac{|f_m^{(1)}| |\varphi_{k-m}| \langle k+n \rangle^s}{|n+m||n-k|}.$$

Since $\langle k+n \rangle^s \leq \langle k-m \rangle^s \langle m+n \rangle^s$, one has

$$\sum_{k \neq n} \sum_{m \neq -n} \frac{|f_m^{(1)}| |\varphi_{k-m}| \langle k+n \rangle^s}{|n+m||n-k|} \leq \sum_{k \neq n} \sum_{m \neq -n} \frac{|f_m^{(1)}| \langle m+n \rangle^s |\varphi_{k-m}| \langle k-m \rangle^s}{|n+m||n-k|}.$$

Now we split the sum into three parts defined by the three sets of summation indices

$$\{\, (k,m) : \ |n-k| > |n|/2 \,\}, \quad \{\, (k,m) : \ |n-k| \leq |n|/2, \ |n+m| > |n|/2 \,\},$$

and

$$\{\, (k,m) : \ |n-k| \leq |n|/2, \ |n+m| \leq |n|/2 \,\}.$$

By the Cauchy-Schwarz inequality

$$\begin{aligned} I :=& \sum_{|n-k| > \frac{|n|}{2}} \sum_{m \neq -n} \frac{|f_m^{(1)}| \langle m+n \rangle^s |\varphi_{k-m}| \langle k-m \rangle^s}{|n+m||n-k|} \\ \leq& \Big(\sum_{|n-k| > \frac{|n|}{2}} \sum_{m \neq -n} \frac{1}{|n-k|^2 |n+m|^2} \Big)^{1/2} \Big(\sum_{|n-k| > \frac{|n|}{2}} \sum_{m \neq -n} |f_m^{(1)}|^2 \langle m+n \rangle^{2s} |\varphi_{k-m}|^2 \langle k-m \rangle^{2s} \Big)^{1/2}. \end{aligned}$$

Hence

$$I \leq \Big(\sum_{|n-k| > \frac{|n|}{2}} \frac{1}{|n-k|^2} \Big)^{1/2} \Big(\sum_{m \neq -n} \frac{1}{|n+m|^2} \Big)^{1/2} \|\varphi\|_s \|f^{(1)}\|_{s;n}.$$

By a similar computation

$$
\begin{aligned}
II :=& \sum_{1\le|n-k|\le\frac{|n|}{2}} \sum_{|n+m|>\frac{|n|}{2}} \frac{|f_m^{(1)}|\langle m+n\rangle^s|\varphi_{k-m}|\langle k-m\rangle^s}{|n+m||n-k|}\\
\le&\Big(\sum_{1\le|n-k|\le\frac{|n|}{2}} \frac{1}{|n-k|^2}\Big)^{1/2}\Big(\sum_{|n+m|>\frac{|n|}{2}} \frac{1}{|n+m|^2}\Big)^{1/2}\|\varphi\|_s\|f^{(1)}\|_{s;n}.
\end{aligned}
$$

Since

$$
\sum_{|l|>\frac{|n|}{2}} \frac{1}{l^2} \le 2\sum_{l>\frac{|n|}{2}} \frac{1}{l^2} \le 2\int_{\frac{|n|}{2}}^{\infty} \frac{1}{l^2}\mathrm{d}l = \frac{4}{|n|}.
$$

It then follows that

$$
\frac{1}{4}I + \frac{1}{4}II \le \frac{1}{2}\big(2\frac{\pi^2}{6}\big)^{1/2}\cdot\big(\frac{4}{|n|}\big)^{1/2}\|\varphi\|_s\|f^{(1)}\|_{s;n} \le \frac{2}{\sqrt{|n|}}\|\varphi\|_s\|f^{(1)}\|_{s;n}
$$

where we used that $\frac{\pi}{\sqrt{3}} < 2$. Now let us turn to the sum

$$
III := \sum_{1\le|n-k|\le\frac{|n|}{2}} \sum_{1\le|n+m|\le\frac{|n|}{2}} \frac{|f_m^{(1)}|\langle m+n\rangle^s|\varphi_{k-m}|\langle k-m\rangle^s}{|n+m||n-k|}
$$

Since $|k-m| \ge 2|n| - |k-n| - |m+n| \ge |n|$ for k, m with $|n-k| \le \frac{|n|}{2}$ and $|m+n| \le \frac{|n|}{2}$ one has

$$
\begin{aligned}
III \le&\Big(\sum_{1\le|n-k|\le\frac{|n|}{2}} \frac{1}{|n-k|^2}\Big)^{1/2}\Big(\sum_{1\le|n+m|\le\frac{|n|}{2}} \frac{1}{|n+m|^2}\Big)^{1/2}\Big(\sum_{|k|\ge|n|}|\varphi_k|^2\langle k\rangle^{2s}\Big)^{1/2}\|f^{(1)}\|_{s;n}\\
\le&2\frac{\pi^2}{6}R_{s;n}(\varphi)\|f^{(1)}\|_{s;n}
\end{aligned}
$$

and hence

$$
\frac{1}{4}III \le R_{s;n}(\varphi)\|f^{(1)}\|_{s;n}.
$$

Altogether we then have proved that

$$
\|g^{(2)}e_n\|_{W^{s,1}} \le \frac{2}{\sqrt{|n|}}\|\varphi\|_s\|f^{(1)}\|_{s;n} + R_{s;|n|}(\varphi)\|f^{(1)}\|_{s;n}. \tag{4.35}
$$

Combining (4.32) and (4.35) we thus have proved that

$$
\left\| T_n \begin{pmatrix} \mathcal{A} & \\ & \end{pmatrix} \mathcal{Q}(\lambda)^{-1}K_n \right\|_{s;n} \le R_s(q,\varphi)\big(\frac{2}{\sqrt{|n|}}\|\varphi\|_s + R_{s;|n|}(\varphi)\big). \tag{4.36}
$$

Similarly one shows that

$$
\|g^{(1)}e_{-n}\|_{W^{s,1}} \le \frac{2}{\sqrt{|n|}}\|\varphi\|_s\|f^{(2)}\|_{s;-n} + R_{s;|n|}(\varphi)\|f^{(2)}\|_{s;-n}
$$

and deduces

$$
\left\| T_n \begin{pmatrix} \mathcal{A} & \\ & \end{pmatrix} \mathcal{Q}(\lambda)^{-1}K_n \right\|_{s;-n} \le R_s(q,\varphi)\big(\frac{2}{\sqrt{|n|}}\|\varphi\|_s + R_{s;|n|}(\varphi)\big). \tag{4.37}
$$

Hence we proved (4.23). $\square$

Decomposing T_n as in Lemma 4.2 the following identities can be verified in a straight forward way.

Lemma 4.4 *Let $(q,\varphi) \in \tilde{H}_c^{s+1}$ with $s \ge 0$ and $\lambda \in \Pi_n$ with $n \in \mathbb{Z}\setminus\{0\}$.*

$$
\begin{aligned}
&\begin{pmatrix} \mathcal{A} & \\ & \end{pmatrix}\mathcal{Q}(\lambda)^{-1}K_n\begin{pmatrix} & \\ \mathcal{B} & \end{pmatrix}\mathcal{Q}(\lambda)^{-1}K_n = 0, &&\begin{pmatrix} & \mathcal{B} \\ & \end{pmatrix}\mathcal{Q}(\lambda)^{-1}K_n\begin{pmatrix} \mathcal{A} & \\ & \end{pmatrix}\mathcal{Q}(\lambda)^{-1}K_n = 0,\\
&\begin{pmatrix} & \\ \mathcal{B} & \end{pmatrix}\mathcal{Q}(\lambda)^{-1}K_n\begin{pmatrix} & \\ \mathcal{B} & \end{pmatrix}\mathcal{Q}(\lambda)^{-1}K_n = 0 &&\begin{pmatrix} & \mathcal{B} \\ & \end{pmatrix}\mathcal{Q}(\lambda)^{-1}K_n\begin{pmatrix} & \mathcal{B} \\ & \end{pmatrix}\mathcal{Q}(\lambda)^{-1}K_n = 0
\end{aligned}
$$

***Lemma* 4.5** *Let $(q,\varphi) \in \tilde{H}_c^{s+1}$ with $s \geq 0$ and $\lambda \in \Pi_n$ with $n \in \mathbb{Z} \setminus \{0\}$.*

(i) There exists an absolute constant $C_0 \geq 1$ so that

$$\|T_n^4\|_{s;\pm n} \leq \frac{C_0}{|n|} R_s^4, \quad R_s \equiv R_s(q,\varphi) := \|\varphi\|_s + \|\cosh(q/2)\|_{s+1} + \|\sinh(q/2)\|_{s+1}.$$

(ii) With $R_{s,|n|}(\varphi)$ given as in (4.24) one has

$$\|T_n^2\|_{s;\pm n} \leq R_s(q,\varphi)\Big(\frac{\|\sinh(q/2)\|_{s+1} + \|\cosh(q/2)\|_{s+1}}{|n|} + \frac{2}{\sqrt{|n|}}\|\varphi\|_s + R_{s;|n|}(\varphi)\Big)$$

(iii) For any $F = \sum_{j=1}^4 f^{(j)} \in H^s(\mathbb{T}_2, \mathbb{C}^4)$ with $f^{(1)} = 0$ the following sharper estimate holds

$$\left\|T_n^2 F\right\|_{s;n} \leq \frac{1}{|n|} R_s^2(q,\varphi)\|F\|_{s;n},$$

(iv) For any $F = \sum_{j=1}^4 f^{(j)} \in H^s(\mathbb{T}_2, \mathbb{C}^4)$ with $f^{(2)} = 0$ one has

$$\left\|T_n^2 F\right\|_{s;-n} \leq \frac{1}{|n|} R_s^2(q,\varphi)\|F\|_{s;-n}.$$

Remark 4.6. It follows from Lemma 4.5(i) that T_n^4 is a $\frac{1}{2}$-contraction for

$$|n| \geq 2C_0 R_s^4.$$

In contrast the estimate of Lemma 4.5(ii) implies that, T_n^2 is a $\frac{1}{2}$-contraction for $|n| \geq N$ where N can be chosen locally uniformly on $\tilde{H}_{\mathbb{C}}^{s+1}$.

Proof. (i) Decomposing T_n as

$$T_n = \begin{pmatrix} \mathcal{A} & \\ & \end{pmatrix} \mathcal{Q}(\lambda)^{-1} K_n + \begin{pmatrix} & \\ \mathcal{B} & \end{pmatrix} \mathcal{Q}(\lambda)^{-1} K_n + \begin{pmatrix} & \mathcal{B} \\ & \end{pmatrix} \mathcal{Q}(\lambda)^{-1} K_n,$$

Lemma 4.4 yields that T_n^4 consists of a sum of terms each either containing $\begin{pmatrix} & \mathcal{B} \\ & \end{pmatrix} \mathcal{Q}(\lambda)^{-1} K_n$ or $\left[\begin{pmatrix} \mathcal{A} & \\ & \end{pmatrix} \mathcal{Q}(\lambda)^{-1} K_n\right]^3$ as a factor. Using (4.16), (4.17), and (4.22) one obtains the claimed estimate $\|T_n^4\|_{s;n} \leq \frac{C_0}{|n|} R_s^4$.

(ii) Note that by Lemma 4.3

$$\|T_n^2\|_{s;\pm n} \leq \|T_n \begin{pmatrix} & \mathcal{B} \\ \mathcal{B} & \end{pmatrix} \mathcal{Q}(\lambda)^{-1} K_n\|_{s;\pm n} + R_s(q,\varphi)\big(\frac{2}{\sqrt{|n|}}\|\varphi\|_s + R_{s;n}(\varphi)\big)$$

and by Lemma 4.2 and Lemma 4.4,

$$\|T_n \begin{pmatrix} & \mathcal{B} \\ \mathcal{B} & \end{pmatrix} \mathcal{Q}(\lambda)^{-1} K_n\|_{s;\pm n} \leq R_s(q,\varphi) \frac{\|\sinh(q/2)\|_{s+1} + \|\cosh(q/2)\|_{s+1}}{|n|}. \tag{4.38}$$

(iii) For any $F = \sum_{j=1}^4 f^{(j)} \in H^s(\mathbb{T}_2, \mathbb{C}^4)$

$$\|T_n^2 F\|_{s;n} \leq \|T_n \begin{pmatrix} & \mathcal{B} \\ \mathcal{B} & \end{pmatrix} \mathcal{Q}(\lambda)^{-1} K_n F\|_{s;n} + \|T_n \begin{pmatrix} \mathcal{A} & \\ & \end{pmatrix} \mathcal{Q}(\lambda)^{-1} K_n F\|_{s;n}.$$

If $f^{(1)} = 0$ then by Lemma 4.3(i)

$$\left\|T_n \begin{pmatrix} \mathcal{A} & \\ & \end{pmatrix} \mathcal{Q}(\lambda)^{-1} K_n F\right\|_{s;n} \leq \frac{1}{2|n|} R_s(q,\varphi)\|\varphi\|_s \|f^{(2)}\|_{s;n}.$$

Hence (4.38) yields (iii)

(iv) Arguing as in the proof of item (iii) one obtains (iv). ☐

Now we go back to the K- and P-equation. Let $(q, \varphi) \in \tilde{H}_c^{s+1}$ be given. Instead of the K-equation (4.14) we consider (4.15) which by the definition of T_n takes the form

$$\mathcal{Q}_0 v = T_n \mathcal{Q}_0(u + v).$$

Solving for $\mathcal{Q}_0 v$ yields

$$(Id - T_n)\mathcal{Q}_0 v = T_n \mathcal{Q}_0 u. \tag{4.39}$$

By Lemma 4.5(i), T_n^4 is a 1/2-contraction for any n with $|n| \geq 2C_0 R_s^4$. It follows that for such n, $(Id - T_n^4)$ and hence $(Id - T_n)$ are invertible where the inverse of $(Id - T_n)$ is given by

$$\hat{T}_n := (Id - T_n)^{-1} = (Id - T_n^4)^{-1}(Id + T_n + T_n^2 + T_n^3).$$

By Lemma 4.2 for any $s \geq 0$ and $|n| \geq 2C_0 R_s^4$

$$\|\hat{T}_n\|_{s;\pm n} \leq 2(1 + R_s + R_s^2 + R_s^3) \leq 2(1 + R_s)^3. \tag{4.40}$$

By (4.39), $Q_0 v$ is given by

$$\mathcal{Q}_0 v = \hat{T}_n T_n \mathcal{Q}_0 u$$

and the P-equation (4.13) becomes

$$\mathcal{Q}(\lambda)u = P_n \mathcal{Q}_0 u + P_n \hat{T}_n T_n \mathcal{Q}_0 u.$$

Since $Id + \hat{T}_n T_n = \hat{T}_n$ one is led to

$$0 = \Big(\mathcal{Q}(\lambda) - P_n \hat{T}_n \mathcal{Q}_0\Big) u.$$

Hence given any $|n| \geq 2C_0 R_s^2(q, \varphi)$, $\lambda \in \Pi_n$ is a periodic eigenvalue of $\mathcal{Q}$ iff $\det(S_n(\lambda)) = 0$ where $S_n(\lambda) \equiv S_n(\lambda, q, \varphi)$ is the map

$$S_n(\lambda) = \Big(\mathcal{Q}(\lambda) - P_n \hat{T}_n \mathcal{Q}_0\Big) P_n : \mathcal{P}_n \to \mathcal{P}_n. \tag{4.41}$$

We now compute the matrix representation of S_n with respect to the basis $[e_{-n}^{(1)}, e_n^{(2)}]$ of $\mathcal{P}_n$. By (4.10), the matrix representation $[Q(\lambda)]$ of $Q(\lambda)$ is given by

$$[\mathcal{Q}(\lambda)] = \begin{pmatrix} \lambda - n\pi & \\ & \lambda - n\pi \end{pmatrix}.$$

and for any $|n| \geq 2C_0 R_s^2(q, \varphi)$ the one of $P_n \hat{T}_n \mathcal{Q}_0 P_n$ is given by

$$\begin{pmatrix} a_n^+(\lambda) & b_n^+(\lambda) \\ b_n^-(\lambda) & a_n^-(\lambda) \end{pmatrix} := \begin{pmatrix} \langle \hat{T}_n \mathcal{Q}_0 e_{-n}^{(1)}, e_{-n}^{(1)} \rangle_c & \langle \hat{T}_n \mathcal{Q}_0 e_n^{(2)}, e_{-n}^{(1)} \rangle_c \\ \langle \hat{T}_n \mathcal{Q}_0 e_{-n}^{(1)}, e_n^{(2)} \rangle_c & \langle \hat{T}_n \mathcal{Q}_0 e_n^{(2)}, e_n^{(2)} \rangle_c \end{pmatrix}. \tag{4.42}$$

For any $\rho \geq 1$, denote by $\tilde{B}_\rho^{s+1}$ the closed ball of radius ρ in $\tilde{H}_c^{s+1}$, centered at 0,

$$\tilde{B}_\rho^{s+1} := \{\ (q, \varphi) \in \tilde{H}_c^{s+1}\ :\ 1 + R_s(q, \varphi) \leq \rho\ \}. \tag{4.43}$$

where we recall that $R_s(q, \varphi) = \|\varphi\|_s + \|\cosh(q/2)\|_{s+1} + \|\sinh(q/2)\|_{s+1}$.

Lemma 4.7 *Let $s \geq 0$, $\rho \geq 1$, and $|n| \geq 2C_0 \rho^4$. Then the following holds:*

(i) A complex number $\lambda \in \Pi_n$ is a periodic eigenvalue of $Q_1 \partial_x + Q_0$ iff $\det S_n(\lambda) = (\lambda - \pi n - a_n(\lambda))^2 - b_n^+(\lambda) b_n^-(\lambda)$, vanishes.

(ii) The functions $a_n^\pm$, $b_n^\pm$ are analytic in $(\lambda, (q, \varphi))$ on $\Pi_n \times \tilde{B}_\rho^{s+1}$. Furthermore $a_n := a_n^+$ coincides with a_n^-, $a_n = a_n^-$, and

$$a_n(\lambda, q, \varphi) = \overline{a_n(\overline{\lambda}, \overline{q}, -\overline{\varphi})}, \quad b_n^-(\lambda, q, \varphi) = \overline{b_n^+(\overline{\lambda}, \overline{q}, -\overline{\varphi})}.$$

Proof. (i) The statement follows from the definition of S_n as mentioned in the discussion above.

(ii) By Lemma 4.5(i) T_n^4 with $|n| \geq 2C_0\rho^4$ is a $1/2$ contraction for any element in $\Pi_n \times \tilde{B}_\rho^{s+1}$. Hence $(Id - T_n^4)^{-1}$ can be expanded in its Neumann series, implying that $a_n(\lambda)$ and $b_n^{\pm}(\lambda)$ can be written as series which converge normally and are analytic on $\Pi_n \times \tilde{B}_\rho^{s+1}$. Note that $Q_0(x) = (\det B(x))^2 = \frac{1}{4}$. Hence $\mathcal{Q}_0(x)$ is invertible for any x and $\hat{T}_n\mathcal{Q}_0$ is invertible as an operator. By the definition of $T_n = \mathcal{Q}_0\mathcal{Q}(\lambda)^{-1}K_n$ it then follows that

$$(\mathcal{Q}_0^{-1}(Id - T_n))^* = \Big(Id - (\mathcal{Q}(\lambda)^{-1}K_n)^*\mathcal{Q}_0^*\Big)(\mathcal{Q}_0^*)^{-1} = (\mathcal{Q}_0^*)^{-1}\Big(Id - \mathcal{Q}_0^*(\mathcal{Q}(\lambda)^{-1}K_n)^*\Big).$$

Using that the adjoints of $\mathcal{Q}_0$, $\mathcal{Q}(\lambda)^{-1}K_n$ with respect to $\langle\cdot,\cdot\rangle_c$ are given by

$$\mathcal{Q}_0(q,\varphi)^* = \mathcal{Q}_0(\overline{q}, -\overline{\varphi}) \quad \text{and} \quad (\mathcal{Q}(\lambda)^{-1}K_n)^* = K_n\mathcal{Q}(\overline{\lambda})^{-1} = \mathcal{Q}(\overline{\lambda})^{-1}K_n$$

one has

$$(\mathcal{Q}_0^{-1}(q,\varphi)(Id - T_n(\lambda,q,\varphi)))^* = (\mathcal{Q}_0^{-1}(\overline{q},-\overline{\varphi})(Id - T_n(\overline{\lambda},\overline{q},-\overline{\varphi}))).$$

Taking the inverse of both sides of the latter identity one gets

$$(\hat{T}_n(\lambda,q,\varphi)\mathcal{Q}_0(q,\varphi))^* = \hat{T}_n(\overline{\lambda},\overline{q},-\overline{\varphi})\mathcal{Q}_0(\overline{q},-\overline{\varphi}). \tag{4.44}$$

Hence $b_n^+(\lambda,q,\varphi) = \overline{b_n^-(\overline{\lambda},\overline{q},-\overline{\varphi})}$ and $a_n^{\pm}(\lambda,q,\varphi) = \overline{a_n^{\pm}(\overline{\lambda},\overline{q},-\overline{\varphi})}$. It remains to prove that $a_n^+ = a_n^-$. For a given linear operator B acting on a $\mathbb{C}$-vector space, denote by $\overline{B}$ its complex conjugate defined by $\overline{B}u := \overline{B\overline{u}}$. Furthermore note that $\begin{pmatrix} Z & \\ & Z \end{pmatrix} e_n^{(1)} = e_n^{(2)}$ and $\begin{pmatrix} Z & \\ & Z \end{pmatrix} e_n^{(2)} = e_n^{(1)}$. Using that $\overline{e_{-n}^{(2)}} = e_n^{(2)}$ and $\langle a,b\rangle_c = \langle \overline{b},\overline{a}\rangle_c$ one then gets

$$\begin{aligned} a_n^+ =& \langle \hat{T}_n\mathcal{Q}_0 e_{-n}^{(1)}, e_{-n}^{(1)}\rangle_c = \langle \hat{T}_n\mathcal{Q}_0 \begin{pmatrix} Z & \\ & Z \end{pmatrix} e_{-n}^{(2)}, \begin{pmatrix} Z & \\ & Z \end{pmatrix} e_{-n}^{(2)}\rangle_c \\ =& \langle e_{-n}^{(2)}, \begin{pmatrix} Z & \\ & Z \end{pmatrix} (\hat{T}_n\mathcal{Q}_0)^* \begin{pmatrix} Z & \\ & Z \end{pmatrix} e_{-n}^{(2)}\rangle_c \\ =& \langle \begin{pmatrix} Z & \\ & Z \end{pmatrix} \overline{(\hat{T}_n\mathcal{Q}_0)^*} \begin{pmatrix} Z & \\ & Z \end{pmatrix} e_n^{(2)}, e_n^{(2)}\rangle_c. \end{aligned}$$

It remains to compute $\begin{pmatrix} Z & \\ & Z \end{pmatrix} \overline{(\hat{T}_n\mathcal{Q}_0)^*} \begin{pmatrix} Z & \\ & Z \end{pmatrix}$. A straightforward computation yields

$$\mathcal{Q}_0(q,\varphi)^* = \mathcal{Q}_0(\overline{q},-\overline{\varphi}) = \begin{pmatrix} Z & \\ & Z \end{pmatrix} \overline{\mathcal{Q}_0(q,\varphi)} \begin{pmatrix} Z & \\ & Z \end{pmatrix}$$

and

$$(\mathcal{Q}(\lambda)^{-1}K_n)^* = \mathcal{Q}(\overline{\lambda})^{-1}K_n = \begin{pmatrix} Z & \\ & Z \end{pmatrix} \overline{\mathcal{Q}(\lambda)^{-1}K_n} \begin{pmatrix} Z & \\ & Z \end{pmatrix}.$$

Hence the adjoint of $\hat{T}_n\mathcal{Q}_0 = \big(Id - \mathcal{Q}_0\mathcal{Q}(\lambda)^{-1}K_n\big)^{-1}\mathcal{Q}_0$ is given by

$$(\hat{T}_n\mathcal{Q}_0)^* = \begin{pmatrix} Z & \\ & Z \end{pmatrix} \overline{(\hat{T}_n\mathcal{Q}_0)} \begin{pmatrix} Z & \\ & Z \end{pmatrix}.$$

This implies that $a_n^+(\lambda) = a_n^-(\lambda)$. $\square$

For a function $f : U \to \mathbb{C}$ on a domain $U \subset X$ of a $\mathbb{C}$-Banach space $(X, \|\cdot\|)$ denote by $|f|_U$ its sup norm,

$$|f|_U := \sup_{\lambda\in U}\|f(\lambda)\|.$$

Lemma 4.8 *Let $(q,\varphi) \in \tilde{H}_c^1$ and $|n| \geq 2C_0R_0^4(q,\varphi)$. Then*

$$|a_n|_{\Pi_n} \leq \frac{1}{|n|}(1 + R_0(q,\varphi))^4R_0^2(q,\varphi) + (1 + R_0(q,\varphi))^4\|\varphi\|_{L^2}R_{0;|n|}(\varphi). \tag{4.45}$$

Furthermore, if in addition $\varphi \in H_{\mathbb{C}}^s$ one has $R_{0;|n|}(\varphi) \leq \frac{1}{\langle n\rangle^s}\|\varphi\|_s$ and hence

$$|a_n|_{\Pi_n} \leq 2(1 + R_0(q,\varphi))^4\Big(\frac{R_0^2(q,\varphi)}{|n|} + \frac{\|\varphi\|_{L^2}\|\varphi\|_s}{\langle n\rangle^s}\Big). \tag{4.46}$$

Proof. Since $\hat{T}_n = Id + \hat{T}_n T_n = Id + T_n + \hat{T}_n T_n^2$ and

$$\mathcal{Q}_0 e_{-n}^{(1)} = \frac{1}{4}(0,\ -\varphi e_{-n},\ \cosh(q/2)e_{-n},\ -\sinh(q/2)e_{-n}), \tag{4.47}$$

one has $\langle \mathcal{Q}_0 e_{-n}^{(1)},\ e_{-n}^{(1)}\rangle_c = 0$. Using that $\hat{T}_n = Id + T_n + \hat{T}_n T_n^2$ we write a_n as $a_n = \Sigma_1 + \Sigma_2$ where

$$\Sigma_1 := \langle T_n \mathcal{Q}_0 e_{-n}^{(1)}, e_{-n}^{(1)}\rangle_c, \quad \Sigma_2 := \langle \hat{T}_n T_n^2 \mathcal{Q}_0 e_{-n}^{(1)}, e_{-n}^{(1)}\rangle_c$$

Substitute $\varphi = \sum_{m\in\mathbb{Z}} \varphi_m e_m$, $\varphi_m \equiv \hat{\varphi}(m) = \int_0^1 \varphi(x) e^{-im\pi x}\,dx$ into $T_n \mathcal{Q}_0 e_{-n}^{(1)} = \mathcal{Q}_0 \mathcal{Q}(\lambda)^{-1} K_n \mathcal{Q}_0 e_{-n}^{(1)}$ to obtain

$$\begin{aligned} T_n \mathcal{Q}_0 e_{-n}^{(1)} =& \mathcal{Q}_0 \mathcal{Q}(\lambda)^{-1} K_n \frac{1}{4} \begin{pmatrix} 0 \\ -\varphi e_{-n} \\ \cosh(q/2)e_{-n} \\ -\sinh(q/2)e_{-n} \end{pmatrix} \\ =& \mathcal{Q}_0 \mathcal{Q}(\lambda)^{-1} \frac{1}{4} \begin{pmatrix} 0 \\ -\sum_{m\neq n} \varphi_{m+n} e_m \\ \cosh(q/2)e_{-n} \\ -\sinh(q/2)e_{-n} \end{pmatrix} \end{aligned}$$

and by (4.10)

$$\mathcal{Q}(\lambda)^{-1} \frac{1}{4} \begin{pmatrix} 0 \\ -\sum_{m\neq n} \varphi_{m+n} e_m \\ \cosh(q/2)e_{-n} \\ -\sinh(q/2)e_{-n} \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 0 \\ -\sum_{m\neq n} \varphi_{m+n} \frac{e_m}{\lambda - n\pi} \\ \frac{1}{\lambda}\cosh(q/2)e_{-n} \\ -\frac{1}{\lambda}\sinh(q/2)e_{-n} \end{pmatrix}.$$

Using trigonometric identities one then concludes

$$T_n \mathcal{Q}_0 e_{-n}^{(1)} = \frac{1}{16\,\lambda} \begin{pmatrix} \cosh(q)e_{-n} \\ -\sinh(q)e_{-n} \\ 0 \\ 0 \end{pmatrix} - \frac{1}{16} \sum_{m\neq n} \frac{\varphi_{m+n}}{\lambda - m\pi} e_m \begin{pmatrix} \varphi \\ 0 \\ -\sinh(q/2) \\ \cosh(q/2) \end{pmatrix}. \tag{4.48}$$

Hence by (4.19), for any $\lambda \in \Pi_n$

$$|\Sigma_1| = |\langle T_n \mathcal{Q}_0 e_{-n}^{(1)}, e_{-n}^{(1)}\rangle_c| \le \frac{1}{16\,|\lambda|} |\widehat{\cosh(q)}(0)| + \frac{1}{16} \sum_{m\neq n} \frac{|\varphi_{m+n}|}{|n-m|} |\varphi_{-(m+n)}|. \tag{4.49}$$

Since $\cosh(q) = \cosh^2(q/2) + \sinh^2(q/2)$ one has by Lemma 4.1

$$\|\cosh(q)\|_{L^2} \le 2\|\cosh(q/2)\|_1^2 + 2\|\sinh(q/2)\|_1^2 \tag{4.50}$$

hence $|\widehat{\cosh(q)}(0)| \le \|\cosh(q)\|_{L^2} \le 2\|\cosh(q/2)\|_1^2 + 2\|\sinh(q/2)\|_1^2$. For the second term in (4.49) split the sum into two parts, $|m-n| > |n|$ and $1 \le |m-n| \le |n|$ to get

$$\frac{1}{16} \sum_{m\neq n} \frac{|\varphi_{m+n}|}{|n-m|} |\varphi_{-(m+n)}| \le \frac{1}{16\,|n|} \|\varphi\|_{L^2}^2 + \frac{1}{16} \sum_{1\le|m-n|\le|n|} |\varphi_{m+n}||\varphi_{-(m+n)}|$$

Using that for $|m-n| \le |n|$ one has $|m+n| = |2n+m-n| \ge 2|n| - |n| = |n|$ and hence

$$\sum_{1\le|m-n|\le|n|} |\varphi_{m+n}||\varphi_{-(m+n)}| \le \|\varphi\|_{L^2} R_{0;|n|}(\varphi).$$

Altogether we thus have shown that

$$|\Sigma_1| \le \frac{R_0^2}{8\,|n|} + \frac{1}{16} \|\varphi\|_{L^2} R_{0;|n|}(\varphi).$$

Towards Σ_2 note that for any vector valued L^2-function f and $1 \le i \le 4$, $|\langle f, e^{(i)}_{-n}\rangle_c| \le \|f\|_{L^2}$. Hence for $f = \hat{T}_n T_n^2 \mathcal{Q}_0 e^{(1)}_{-n}$,

$$|\Sigma_2| = |\langle \hat{T}_n T_n^2 \mathcal{Q}_0 e^{(1)}_{-n}, e^{(1)}_{-n}\rangle_c| \le \|\hat{T}_n T_n^2 \mathcal{Q}_0 e^{(1)}_{-n}\|_{L^2}.$$

Hence by (4.40)

$$|\Sigma_2| \le \|\hat{T}_n T_n^2 \mathcal{Q}_0 e^{(1)}_{-n}\|_{L^2} \le 2\,(1+R_0)^3\,\|T_n^2 \mathcal{Q}_0 e^{(1)}_{-n}\|_{L^2}. \tag{4.51}$$

Furthermore, by (4.48)

$$T_n^2 \mathcal{Q}_0 e^{(1)}_{-n} = \frac{1}{16\,\lambda} T_n \begin{pmatrix} \cosh(q) e_{-n} \\ -\sinh(q) e_{-n} \\ 0 \\ 0 \end{pmatrix} - \frac{1}{64\,\lambda} \sum_{m \ne n} \frac{\varphi_{m+n}}{\lambda - m\pi} e_m \begin{pmatrix} -\sinh(q) \\ \cosh(q) \\ 0 \\ 0 \end{pmatrix} - T_n \frac{1}{16} \sum_{m \ne n} \frac{\varphi_{m+n}}{\lambda - m\pi} \varphi e^{(1)}_m. \tag{4.52}$$

We will now estimate the three terms on the right hand side in the latter identity separately. One easily checks using Lemma 4.2 that for any $\lambda \in \Pi_n$

$$\|\frac{1}{16\,\lambda} T_n \begin{pmatrix} \cosh(q) e_{-n} \\ -\sinh(q) e_{-n} \\ 0 \\ 0 \end{pmatrix}\|_{L^2} \le \frac{1}{16|n|} R_0 \big(\|\cosh(q)\|_{L^2} + \|\sinh(q)\|_{L^2}\big).$$

For the second term on the right hand side of the identity (4.52) one has for any $\lambda \in \Pi_n$

$$\|\frac{1}{64\,\lambda} \sum_{m \ne n} \frac{\varphi_{m+n}}{\lambda - m\pi} e_m \begin{pmatrix} -\sinh(q) \\ \cosh(q) \\ 0 \\ 0 \end{pmatrix}\|_{L^2} \le \frac{1}{64|n|} (\|\sinh(q)\|_{L^2} + \|\cosh(q)\|_{L^2}) \|\varphi\|_{L^2}.$$

For the last term in (4.52) one has by Lemma 4.2

$$\|T_n \frac{1}{16} \sum_{m \ne n} \frac{\varphi_{m+n}}{\lambda - m\pi} \varphi e^{(1)}_m\|_{L^2} \le R_0 \frac{1}{16} \|\varphi\|_{L^2} \sqrt{\sum_{m \ne n} \frac{|\varphi_{m+n}|^2}{|n-m|^2}}$$

where by arguing as above

$$\Big(\sum_{m \ne n} \frac{|\varphi_{m+n}|^2}{|n-m|^2}\Big)^{1/2} \le \frac{1}{|n|} \|\varphi\|_{L^2} + R_{0;|n|}(\varphi).$$

Altogether we have proved

$$\|T_n^2 \mathcal{Q}_0 e^{(1)}_{-n}\|_{L^2} \le \frac{1}{4\,|n|} R_0 \left(\|\varphi\|^2_{L^2} + \frac{1}{2} (\|\cosh(q)\|_{L^2} + \|\sinh(q)\|_{L^2}) \right) + \frac{1}{16} R_0 \|\varphi\|_{L^2} R_{0;|n|}(\varphi).$$

By Lemma 4.1 $\|\sinh(q)\|_{L^2} \le 4 \|\cosh(q/2)\|_1 \|\sinh(q/2)\|_1$ and with (4.50) one obtains

$$\left(\|\varphi\|^2_{L^2} + \frac{1}{2} (\|\cosh(q)\|_{L^2} + \|\sinh(q)\|_{L^2}) \right) \le \big(\|\varphi\|^2_{L^2} + (\|\cosh(q/2)\|_1 + \|\sinh(q/2)\|_1)^2 \big) \le R_0^2.$$

Hence by (4.51),

$$|\Sigma_2| \le \frac{1}{2|n|} (1+R_0)^3 R_0^3 + (1+R_0)^3 R_0 \|\varphi\|_{L^2} R_{0;|n|}(\varphi).$$

Combining the estimates for Σ_1 and Σ_2 yields (4.45). The estimate (4.46) immediately follows from (4.45).

□

Next we estimate $b_n^+(\lambda)$, $b_n^-(\lambda)$, introduced in (4.42)

Lemma 4.9 *Let $s \geq 0$, $(q,\varphi) \in \tilde{H}_c^{s+1}$ and $\lambda \in \Pi_n$ with $|n| \geq 2C_0R_s^4$. Then the following holds:*

$$\langle n\rangle^s |b_n^{\pm}(\lambda) \mp \frac{1}{4}\widehat{\varphi}(\mp 2n)|_{\Pi_n} \leq \frac{1}{2|n|}\left(1 + R_s(q,\varphi)\right)^6, \tag{4.53}$$

$$|b_n^+ b_n^-|_{\Pi_n} \leq \frac{|\widehat{\varphi}(-2n)|^2 + |\widehat{\varphi}(2n)|^2}{16} + \frac{1}{2n^2}\left(1 + R_0(q,\varphi)\right)^{12}, \tag{4.54}$$

and

$$\langle n\rangle^{2s} |b_n^+ b_n^-|_{\Pi_n} \leq \frac{1}{16}\|\varphi\|_s^2 + \frac{1}{2n^2}\big(1 + R_s(q,\varphi)\big)^{12}. \tag{4.55}$$

Furthermore

$$\sqrt{\sum_{n \geq 2C_0(1+R_s)^4+1} \langle n\rangle^{2s} 6 |b_n^+ b_n^-|_{\Pi_n}} \leq \|\varphi\|_s + 2\big(1 + R_s(q,\varphi)\big)^4. \tag{4.56}$$

Proof. We begin by proving the estimate (4.53) for $b_n^-(\lambda)$ for n with $|n| \geq 2C_0R_s^4$. By the definition (4.42), $b_n^-(\lambda) = \langle \hat{T}_n \mathcal{Q}_0 e_{-n}^{(1)}, e_n^{(2)}\rangle_c$. Arguing as in the proof of Lemma 4.8 one gets for any $\lambda \in \Pi_n$

$$b_n^-(\lambda) = \langle \mathcal{Q}_0 e_{-n}^{(1)}, e_n^{(2)}\rangle_c + \langle T_n \mathcal{Q}_0 e_{-n}^{(1)}, e_n^{(2)}\rangle_c + \langle \hat{T}_n T_n^2 \mathcal{Q}_0 e_{-n}^{(1)}, e_n^{(2)}\rangle_c.$$

By (4.47) one has

$$\langle \mathcal{Q}_0 e_{-n}^{(1)}, e_n^{(2)}\rangle_c = -\frac{1}{4}\int_0^1 \varphi e_{-n} \cdot e_{-n}\,\mathrm{d}x = -\frac{1}{4}\hat{\varphi}(2n).$$

By (4.48) and (4.7), for any $\lambda \in \Pi_n$

$$\begin{aligned}|\langle T_n \mathcal{Q}_0 e_{-n}^{(1)}, e_n^{(2)}\rangle_c| =& \frac{1}{16\,|\lambda|}|\widehat{\sinh(q)}(2n)| \leq \frac{1}{16|\lambda|\langle 2n\rangle^s}\|\sinh(q)\|_s \\ \leq& \frac{1}{4\,|n|\langle 2n\rangle^s}\|\sinh(q/2)\|_{s+1}\|\cosh(q/2)\|_{s+1}.\end{aligned}$$

By (4.40)

$$\langle n\rangle^s |\langle \hat{T}_n T_n^2 \mathcal{Q}_0 e_{-n}^{(1)}, e_n^{(2)}\rangle_c| \leq \|\hat{T}_n T_n^2 \mathcal{Q}_0 e_{-n}^{(1)}\|_{s;n} \leq 2(1+R_s)^3 \|T_n^2 \mathcal{Q}_0 e_{-n}^{(1)}\|_{s;n}.$$

Since by (4.47) the first entry of $\mathcal{Q}_0 e_{-n}^{(1)}$ vanishes one can apply Lemma 4.5(iii) to obtain

$$\|T_n^2 \mathcal{Q}_0 e_{-n}^{(1)}\|_{s;n} \leq \frac{1}{|n|}R_s^2 \|\mathcal{Q}_0 e_{-n}^{(1)}\|_{s;n}.$$

Since by the formula (4.47) for $\mathcal{Q}_0 e_{-n}^{(1)}$ one has $\|\mathcal{Q}_0 e_{-n}^{(1)}\|_{s;n} \leq \frac{1}{4}R_s$ the claimed estimate of $b_n^-(\lambda)$ of (4.53) follows. The estimate for $b_n^+(\lambda)$ is proved in a similar fashion.

To prove (4.54) and (4.55) use that $|ab| \leq \frac{1}{2}(|a|^2 + |b|^2)$ to obtain for $\lambda \in \Pi_n$

$$\begin{aligned}|b_n^+ b_n^-| \leq& \Big(|b_n^+ - \frac{1}{4}\widehat{\varphi}(-2n)| + |\frac{1}{4}\widehat{\varphi}(-2n)|\Big)\Big(|b_n^- + \frac{1}{4}\widehat{\varphi}(2n)| + |\frac{1}{4}\widehat{\varphi}(2n)|\Big) \\ \leq& |b_n^+ - \frac{1}{4}\widehat{\varphi}(-2n)|^2 + |\frac{1}{4}\widehat{\varphi}(-2n)|^2 + |b_n^- + \frac{1}{4}\widehat{\varphi}(2n)|^2 + |\frac{1}{4}\widehat{\varphi}(2n)|^2.\end{aligned}$$

Hence by (4.53), for any $\lambda \in \Pi_n$,

$$|b_n^+ b_n^-| \leq \frac{1}{16}\Big(|\widehat{\varphi}(-2n)|^2 + |\widehat{\varphi}(2n)|^2\Big) + \frac{1}{2n^2\langle n\rangle^{2s}}\left(1 + R_s\right)^{12}. \tag{4.57}$$

For $s = 0$, this yields (4.54) and for $s \geq 0$ arbitrary (4.55). Finally since

$$\sum_{n \geq 2C_0(1+R_s)^4+1} \langle n\rangle^{2s}|\widehat{\varphi}(-2n)|^2 + \langle n\rangle^{2s}|\widehat{\varphi}(2n)|^2 \leq \|\varphi\|_s^2$$

and since $C_0 \geq 1$ and therefore

$$\sum_{n \geq 2C_0(1+R_s)^4+1} \frac{1}{n^2} \leq \int_{2C_0(1+R_s)^4}^{\infty} \frac{1}{x^2}\,\mathrm{d}x = \frac{1}{2C_0(1+R_s)^4}$$

(4.56) follows from (4.57). $\square$

***Theorem* 4.10** *Let $s \geq 0$ and $(q,p) \in H_c^{s+1}$. Then there exists $N_1 \geq 2C_0(1+R_s)^4 + 1$ such that for any $|n| \geq N_1$, the determinant $\det S_n = (\lambda - \pi n - a_n(\lambda))^2 - b_n^+(\lambda) b_n^-(\lambda)$ has, counted with multiplicity, exactly two roots $\lambda_n^\pm$ in Π_n. They are contained in the discs $D_n \subset \Pi_n$ where we recall that by (3.1)*

$$D_n = \{ \lambda \in \mathbb{C} : \ |\lambda - \pi n| < \pi/3 \}.$$

Furthermore, $\gamma_n = \lambda_n^+ - \lambda_n^-$, satisfy

$$|\gamma_n|^2 \leq 6 |b_n^+ b_n^-|_{\Pi_n} \quad n \geq N_1 \tag{4.58}$$

and

$$\left(\sum_{n \geq N_1} \langle n \rangle^{2s} |\gamma_n(q,p)|^2 \right)^{1/2} \leq \|\varphi(q,p)\|_s + 2\,(1 + R_s(q,p))^4 \tag{4.59}$$

where by a slight abuse of terminology, $R_s(q,p) \equiv R_s(q, \varphi(q,p)) = \|\varphi\|_s + \|\sinh(q/2)\|_{s+1} + \|\cosh(q/2)\|_{s+1}$.

Remark 4.11. Recall that by the reciprocity law, for any $n \geq 0$,

$$\frac{1}{16\lambda_{-n}^-(q,p)} - \frac{1}{16\lambda_{-n}^+(q,p)} = \lambda_n^+(-q,p) - \lambda_n^-(-q,p) = \gamma_n(-q,p).$$

Hence (4.58) applied to $(-q,p)$ leads to the estimate

$$\left| \frac{1}{16\lambda_{-n}^-(q,p)} - \frac{1}{16\lambda_{-n}^+(q,p)} \right|^2 \leq 6 |b_n^+ b_n^-|_{\Pi_n, -q, p}, \quad n \geq N_1$$

Proof. By assumption $(q,\varphi) \in \tilde{H}^{s+1}$. According to Lemma 4.8 and 4.9 there exists $N_1 \geq 2C_0(1+R_0)^4+1$ such that for any $|n| \geq N_1$

$$|a_n|_{\Pi_n}, |b_n^+ b_n^-|_{\Pi_n} \leq \frac{\pi}{48}.$$

Hence, for any $|n| \geq N_1$ and $\lambda \in \Pi_n$

$$|\det S_n(\lambda) - (\lambda - n\pi - a_n(\lambda))^2| \leq |b_n^+(\lambda) b_n^-(\lambda)|^2 \leq \left(\frac{\pi}{48} \right)^2$$

and

$$\inf_{\lambda \in \partial D_n} |\lambda - n\pi - a_n(\lambda)|^2 \geq \left| \frac{\pi}{3} - \sup_{\lambda \in \partial D_n} |a_n(\lambda)| \right|^2 > \left(\frac{\pi}{48} \right)^2.$$

As $\det S_n(\lambda)$ and $(\lambda - n\pi - a_n(\lambda))^2$ are both analytic on Π_n, by Rouché's Theorem, they have the same number of roots in D_n when counted with multiplicities. By the same argument, one shows that $(\lambda - n\pi - a_n(\lambda))^2$ and $(\lambda - n\pi)^2$ have the same number of roots in D_n when counted with multiplicities. Hence $\det S_n(\lambda)$ has two roots in D_n. By choosing N_1 larger than the integer N in Lemma 3.11 (Counting Lemma), it follows that these two roots are precisely the periodic eigenvalues λ_n^+ and λ_n^-.

To prove the claimed estimate for the gaps $\gamma_n = \lambda_n^+ - \lambda_n^-$ we write

$$\det S_n = (\lambda - \pi n - a_n)^2 - b_n^+ b_n^- = g_+ g_- \tag{4.60}$$

where

$$g_\pm = \lambda - n\pi - a_n \pm \sigma_n, \quad \sigma_n = \sqrt{b_n^+ b_n^-}, \tag{4.61}$$

where the choice of the branch of the root is immaterial. Each root $\xi_n \in D_n$ of $\det S_n$ is either a root of g_+ or g_- and hence is of the form $\xi_n = n\pi + a_n(\xi_n) \pm \sigma(\xi_n)$. It then follows that

$$|\lambda_n^+ - \lambda_n^-| \leq |a_n(\lambda_n^+) - a_n(\lambda_n^-)| + |\sigma_n(\lambda_n^+)| + |\sigma_n(\lambda_n^-)| \leq |\partial_\lambda a_n|_{D_n} |\lambda_n^+ - \lambda_n^-| + 2|\sigma_n|_{\Pi_n}.$$

Since $dist(D_n, \partial\Pi_n) \geq \pi/6$, we obtain from Cauchy's estimate

$$|\partial_\lambda a_n|_{D_n} \leq \frac{|a_n|_{\Pi_n}}{\pi/6} \leq \frac{1}{8},$$

which implies that $\frac{7}{8}|\lambda_n^+ - \lambda_n^-| \le 2\left|\sqrt{b_n^+ b_n^-}\right|_{\Pi_n}$ and therefore

$$|\lambda_n^+ - \lambda_n^-|^2 \le 6|b_n^+ b_n^-|_{\Pi_n}.$$

By Lemma 4.9 one then concludes that

$$\sqrt{\sum_{n\ge N_1} \langle n\rangle^{2s}|\gamma_n|^2} \le \|\varphi\|_s + 2\,(1+R_s)^4\,.$$

Together with Lemma 2.14 (reciprocity in λ) this yields (4.58) and (4.59). □

Remark 4.12. Assume that $(q,p) \in H^1_r$. Since q,p are then real valued, one has $-\overline{\varphi} = \varphi$ and hence by Lemma 4.7

$$b_n^-(\lambda,q,\varphi) = \overline{b_n^+(\lambda,q,\varphi)}, \quad a_n(\lambda,q,\varphi) \in \mathbb{R}, \quad \forall\, \lambda \in \mathbb{R}^*.$$

Furthermore, by Lemma 4.9 ($s=0$)

$$|b_n^+(\lambda_n^\pm) - \frac{1}{4}\hat{\varphi}(-2n)| = O(\frac{1}{n})$$

and hence

$$\sigma_n(\lambda_n^\pm) = |b_n^+(\lambda_n^\pm)| = \frac{1}{4}|\hat{\varphi}(-2n)| + O(\frac{1}{n}).$$

On the other side, by the definition of $a_n(\lambda)$,

$$a_n(\lambda) = \langle (Id - T_n(\lambda))^{-1}\mathcal{Q}_0 e_{-n}^{(1)}, e_{-n}^{(1)}\rangle_c.$$

Expanding $(Id - T_n(\lambda))^{-1}$ in the form

$$(Id - T_n(\lambda))^{-1} = Id + \sum_{k=1}^{3} T_n(\lambda)^k + T_n(\lambda)^4 (Id - T_n(\lambda))^{-1}$$

one obtains from Lemma 4.5 and (4.47)

$$a_n(\lambda_n^\pm) = \sum_{k=1}^{3}\langle T_n(\lambda_n^\pm)^k \mathcal{Q}_0 e_{-n}^{(1)}, e_{-n}^{(1)}\rangle_c + O(\frac{1}{n}).$$

By (4.60)-(4.61), there exists $\rho_n^\pm \in \{1,-1\}$ so that

$$\lambda_n^\pm = n\pi + a_n(\lambda_n^\pm) + \rho_n^\pm \sigma_n(\lambda_n^\pm).$$

It implies that

$$\lambda_n^+ - \lambda_n^- = \sum_{k=1}^{3}\langle (T_n(\lambda_n^+)^k - T_n(\lambda_n^-)^k)\mathcal{Q}_0 e_{-n}^{(1)}, e_{-n}^{(1)}\rangle_c + (\rho_n^+ - \rho_n^-)\frac{1}{4}|\hat{\varphi}(-2n)| + O(\frac{1}{n}).$$

4.2 Adapted Fourier coefficients

For $|n|$ sufficiently large the 2×2 matrix S_n contains all the information about the n-th periodic eigenvalues of a potential. In order to characterize their asymptotics for $|n| \to \infty$ in terms of the regularity of the potential, we analyze $S_n(\lambda)$, $\lambda \in \Pi_n$, further. We will prove that the diagonal of $S_n(\lambda)$ vanishes at a unique point

$$\lambda = \sigma_n(q,\varphi).$$

These values will be used to locally define a real analytic perturbation of the Fourier transform which allows to characterize the regularity of potentials mentioned above. First we need to establish some auxiliary results.

***Lemma* 4.13** *Let $s > 0$ and $\rho > 0$. Then for any $(q,\varphi) \in \tilde{B}_\rho^{s+1}$ and $|n| \ge \max(2C_0\rho^4, \sqrt[s]{96\rho^6}, 96\rho^6)$, there is a unique analytic function $\sigma_n : \tilde{B}_\rho^{s+1} \to \mathbb{C}$ such that*

(i) $\sigma_n(q,\varphi) = n\pi + a_n(\sigma_n(q,\varphi),(q,\varphi)) \quad \forall\ (q,\varphi) \in \tilde{B}^{s+1}_\rho$.

(ii) $\sup_{(q,\varphi)\in\tilde{B}^{s+1}_\rho} |\sigma_n(q,\varphi) - n\pi| \le \frac{\pi}{48}$,

(iii) $\sigma_n(q,\varphi) \in \mathbb{R}$ *for any real valued* $(q,\varphi) \in \tilde{B}^{s+1}_\rho$.

Proof. For any given $|n| \ge \max(2C_0\rho^4, \sqrt[s]{48\rho^6}, 96\rho^6)$, consider the map T

$$T\sigma := n\pi + a_n(\sigma(\cdot),\cdot)$$

with domain of definition

$$E := \{\ \sigma : \tilde{B}^{s+1}_\rho \to D'_n \ :\ \sigma \text{ real analytic }\},$$

and $D'_n = \{\ \lambda \in \mathbb{C}\ :\ |\lambda - n\pi| \le \pi/48\ \} \subset D_n$. E is obviously not empty since the constant function $\sigma \equiv n\pi$ is in E. Note that by the definition of $\tilde{B}^{s+1}_\rho$, the assumed lower bound for $|n|$, and Lemma 4.8,

$$|a_n|_{\Pi_n\times\tilde{B}^{s+1}_\rho} \le 2(1+R_0)^4\Big(\frac{R_0^2}{|n|} + \frac{\|\varphi\|_{L^2}\|\varphi\|_s}{\langle n\rangle^s}\Big) \le 2\rho^4\Big(\frac{\rho^2}{96\rho^6} + \frac{\rho^2}{96\rho^6}\Big) \le \frac{\pi}{48}$$

implying that T maps E into E. Endow E with the metric $d(\sigma_1,\sigma_2) = |\sigma_1 - \sigma_2|_{\tilde{B}^{s+1}_\rho}$. Then E is complete. We claim that T is a contraction. Indeed

$$d(T(\sigma_1),T(\sigma_2)) = |T(\sigma_1) - T(\sigma_2)|_{\tilde{B}^{s+1}_\rho} \le |\partial_\lambda a_n|_{D'_n\times\tilde{B}^{s+1}_\rho} d(\sigma_1,\sigma_2) \le \frac{1}{23} d(\sigma_1,\sigma_2)$$

as by Cauchy's estimate

$$|\partial_\lambda a_n|_{D'_n\times\tilde{B}^{s+1}_\rho} \le \frac{|a_n|_{\Pi_n\times\tilde{B}^{s+1}_\rho}}{\operatorname{dist}(D'_n,\partial\Pi_n)} \le \frac{\pi/48}{\pi/2 - \pi/48} = \frac{1}{23}.$$

Hence T admits a unique fixed point in E, denoted by σ_n. By construction, σ_n satisfies items (i)-(ii). Furthermore item (iii) holds since by the uniqueness of σ_n and Lemma 4.7(ii) one has $\sigma_n(\overline{q},-\overline{\varphi}) = \overline{\sigma_n(q,\varphi)}$. $\square$

Let $s > 0$ and $\rho > 0$. Then for any $|n| \ge \max(2C_0\rho^4, \sqrt[s]{96\rho^6})$ and $(q,\varphi) \in \tilde{B}^{s+1}_\rho$

$$S_n(\sigma_n(q,\varphi),q,\varphi) = \begin{pmatrix} 0 & -b_n^+(\sigma_n(q,\varphi),q,\varphi) \\ -b_n^-(\sigma_n(q,\varphi),q,\varphi) & 0 \end{pmatrix}.$$

By Lemma 4.9 we know that $b_n^+(\lambda)$ is close to $\frac{1}{4}\widehat{\varphi}(-2n)$ and $b_n^-(\lambda)$ is close to $-\frac{1}{4}\widehat{\varphi}(2n)$.

For any given $s > 0$ define the perturbed Fourier series $\mathcal{F}_{s,\rho}(q,\varphi) \in H^s_{\mathbb{C}}$ for $(q,\varphi) \in \tilde{B}^{s+1}_\rho$ as follows

$$\mathcal{F}_{s,\rho}(q,\varphi) := \sum_{|n|\le M_{s,\rho}+1} \varphi_n e_n + \sum_{n>M_{s,\rho}+1} 4b_n^+(\sigma_n(q,\varphi),q,\varphi)e_{-2n} - 4b_n^-(\sigma_n(q,\varphi),q,\varphi)e_{2n} \tag{4.62}$$

where

$$M_{s,\rho} := max(2C_0\rho^4, \sqrt[s]{96\rho^6}, 96\rho^6, 2^{20+2s}\rho^{10}). \tag{4.63}$$

The choice of $M_{s,\rho}$ ensures that $\mathcal{F}_{s,\rho}$ is a local diffeomorphism (see proof of Lemma 4.14 below)

Furthermore we introduce

$$\Phi_{s,\rho} : \tilde{B}^{s+1}_\rho \to \tilde{H}^{s+1}_c, (q,\varphi) \mapsto (q, \mathcal{F}_{s,\rho}(q,\varphi)).$$

***Lemma* 4.14** *Let $\rho \ge 1$ and $s > 0$, Then $\Phi_{s,\rho}$ is a real analytic diffeomorphism*

$$\Phi_{s,\rho} : \tilde{B}^{s+1}_\rho \to \Phi^s_\rho(\tilde{B}^{s+1}_\rho) \subset \tilde{H}^{s+1}_c.$$

such that

$$\frac{\|\varphi\|_s}{2} \le \|\mathcal{F}_{s,\rho}(q,\varphi)\|_s \le 2\|\varphi\|_s, \qquad \forall\ \ (q,\varphi) \in \tilde{B}^{s+1}_\rho$$

implying that $\tilde{B}^{s+1}_{\rho/2} \subset \Phi_{s,\rho}(\tilde{B}^{s+1}_\rho)$. Moreover

$$\sup_{(q,\varphi)\in\tilde{B}^s_\rho} \|\partial_\varphi\mathcal{F}_{s,\rho}(q,\varphi) - Id_{H^s_{\mathbb{C}}}\|_s \le \frac{1}{4}.$$

Proof. For any $|n| \geq M_{s,\rho} \geq 2C_0\rho^4$, σ_n maps $\tilde{B}^{s+1}_{2\rho}$ into Π_n (cf. Lemma 4.13(ii)) and $b^{\pm}_n(\sigma_n(q,\varphi),q,\varphi)$ is well defined for $(q,\varphi) \in \tilde{B}^{s+1}_{2\rho}$. Furthermore by (4.53)

$$\langle n\rangle^s |4b^+_n(\sigma_n(q,\varphi),q,\varphi) - \hat{\varphi}(-2n)|_{\tilde{B}^s_{2\rho}} \leq \langle n\rangle^s |4b^+_n - \hat{\varphi}(-2n)|_{\Pi_n\times\tilde{B}^s_{2\rho}} \leq \frac{2}{|n|}(2\rho)^6, \tag{4.64}$$

$$\langle n\rangle^s |4b^-_n(\sigma_n(q,\varphi),q,\varphi) + \hat{\varphi}(2n)|_{\tilde{B}^s_{2\rho}} \leq \langle n\rangle^s |4b^-_n + \hat{\varphi}(2n)|_{\Pi_n\times\tilde{B}^s_{2\rho}} \leq \frac{2}{|n|}(2\rho)^6. \tag{4.65}$$

Hence the map $\mathcal{F}_{s,\rho}$ is defined on $\tilde{B}^{s+1}_{2\rho}$ and takes values in $H^s_{\mathbb{C}}$. Moreover, by the definition of $\mathcal{F}_{s,\rho}$ and (4.64)-(4.65)

$$\begin{aligned}\sup_{(q,\varphi)\in\tilde{B}^{s+1}_{2\rho}} \|\mathcal{F}_{s,\rho}(q,\varphi)-\varphi\|^2_s \leq & \sum_{n>M_{s,\rho}+1} \langle 2n\rangle^{2s}|4b^+_n(\sigma(q,\varphi),q,\varphi)-\hat{\varphi}(-2n)|^2_{\tilde{B}^{s+1}_{2\rho}} + \langle 2n\rangle^{2s}|4b^-_n(\sigma(q,\varphi),q,\varphi)+\hat{\varphi}(2n)|^2_{\tilde{B}^{s+1}_{2\rho}}\\ \leq & 2^{2s}\sum_{n>M_{s,\rho}+1} \frac{8}{n^2}(2\rho)^{12} \leq 2^{16+2s}\frac{\rho^{12}}{M_{s,\rho}} \leq \frac{\rho^2}{16}.\end{aligned}$$

By Cauchy's estimate applied to $\mathcal{F}_{s,\rho}(q,\cdot)$ on $\tilde{B}^{s+1}_{\rho}$

$$\sup_{(q,\varphi)\in\tilde{B}^{s+1}_{\rho}} \|\partial_\varphi \mathcal{F}_{s,\rho}(q,\varphi) - Id_{H^s_{\mathbb{C}}}\|_s \leq \frac{1}{\rho}\sup_{(q,\varphi)\in\tilde{B}^{s+1}_{2\rho}} \|\mathcal{F}_{s,\rho}(q,\varphi)-\varphi\|_s \leq \frac{1}{4}.$$

Hence $d_\varphi\mathcal{F}_{s,\rho}(q,\varphi) : H^s_{\mathbb{C}} \to H^s_{\mathbb{C}}$ is invertible for all $(q,\varphi) \in \tilde{B}^{s+1}_{\rho}$ and so is $d\Phi_{s,\rho} = \begin{pmatrix} Id & 0 \\ \partial_q\mathcal{F}_{s,\rho} & \partial_\varphi\mathcal{F}_{s,\rho}\end{pmatrix}$. We thus have proved that for any $s > 0$, $\Phi_{s,\rho} : \tilde{B}^{s+1}_{\rho} \to \tilde{H}^{s+1}_c$ is a local diffeomorphism. Furthermore one has

$$|\|\mathcal{F}_{s,\rho}(q,\varphi)\|_s - \|\varphi\|_s| \leq \|\mathcal{F}_{s,\rho}(q,\varphi)-\varphi\|_s \leq sup_{(q,\varphi)\in\tilde{B}^{s+1}_{\rho}}\|\partial_\varphi\mathcal{F}_{s,\rho}(q,\varphi) - Id_{H^s_{\mathbb{C}}}\|_s\|\varphi\| < \frac{1}{4}\|\varphi\|.$$

Hence

$$\frac{\|\varphi\|_s}{2} \leq \|\mathcal{F}_{s,\rho}(q,\varphi)\|_s \leq 2\|\varphi\|_s, \quad \forall\ (q,\varphi) \in \tilde{B}^{s+1}_{\rho}.$$

To see that $\Phi_{s,\rho} : \tilde{B}^{s+1}_{\rho} \to \tilde{H}^{s+1}_c$ is one-to-one note that for any $(q,\varphi_1),(q,\varphi_2) \in \tilde{B}^{s+1}_{\rho}$

$$\|\mathcal{F}_{s,\rho}(q,\varphi_1) - \mathcal{F}_{s,\rho}(q,\varphi_2) - (\varphi_1-\varphi_2)\|_s \leq \sup_{(q,\varphi)\in\tilde{B}^{s+1}_{\rho}} \|\partial_\varphi\mathcal{F}_{s,\rho} - Id_{H^1_{\mathbb{C}}}\|_s\|\varphi_1-\varphi_2\|_s \leq \frac{1}{4}\|\varphi_1-\varphi_2\|_s.$$

Thus if $\mathcal{F}_{s,\rho}(q,\varphi_1) = \mathcal{F}_{s,\rho}(q,\varphi_2)$, one has $\|\varphi_1-\varphi_2\|_s \leq \frac{1}{4}\|\varphi_1-\varphi_2\|$ which implies $\varphi_1 = \varphi_2$. □

Denote by LFG^s_c and RFG^s_c the following subsets of H^s_c

$$LFG^s_c := \{\ (q,p) \in H^s_c\ :\ (q,p) \text{ left sided finite gap potential }\}$$

and

$$RFG^s_c := \{\ (q,p) \in H^s_c\ :\ (q,p) \text{ right sided finite gap potential }\}.$$

Theorem 4.15 *(i) For any $s \in \mathbb{R}_{\geq 1}$ the sets LFG^s_c and RFG^s_c are both dense in H^s_c.*

(ii) For any $s \in \mathbb{R}_{\geq 1}$ the sets $LFG^s_c \cap H^s_r$ and $RFG^s_c \cap H^s_r$ are dense in H^s_r.

Proof. (i) Since $\tilde{H}^{s+1}_c$ and H^{s+1}_c are isomorphic (cf. (4.8)) and $\tilde{H}^{s+1}_c$ is dense in $\tilde{H}^s_c$ it suffices to prove that for any $s \in \mathbb{R}_{\geq 1}$ and $\rho \geq 1$, the sets

$$\tilde{\mathcal{G}}^{s+1}_{\rho,\text{right}} := \{\ (q,\varphi) \in \tilde{B}^{s+1}_{\rho}\ :\ (q,\varphi) \text{ is right sided finite gap potential }\}$$

and

$$\tilde{\mathcal{G}}^{s+1}_{\rho,\text{left}} := \{\ (q,\varphi) \in \tilde{B}^{s+1}_{\rho}\ :\ (q,\varphi) \text{ is left sided finite gap potential }\}$$

are dense in $\tilde{B}^{s+1}_{\rho}$. By a slight abuse of terminology we say that $(q,\varphi) = (q, Pp+q_x)$ is a right or left sided finite gap potential if (q,p) is such a potential. Let us first prove that $\tilde{\mathcal{G}}^{s+1}_{\rho,\text{right}}$ is dense in $\tilde{B}^{s+1}_{\rho}$. For any

$M \in \mathbb{Z}_{\geq 1}$, denote by $\mathcal{G}_{s,M}$ the closed subspace of $H^s_{\mathbb{C}}$ spanned by $e_{2k} = e^{\mathrm{i}2k\pi x}$, $|k| \leq M$. Then $\mathcal{G}_{s,M}$ is an increasing sequence of subspaces of $H^1_{\mathbb{C}}$ and $\bigcup_{M \geq M_0} \mathcal{G}_{s,M}$ is dense in $H^s_{\mathbb{C}}$ and hence $\bigcup_{M \geq M_0} H^{s+1}_{\mathbb{C}} \times \mathcal{G}_{s,M}$ is dense in $\tilde{H}^{s+1}_c$. Here $M_0 = \max(M_{s,\rho}, N_1)$ where $M_{s,\rho}$ is given by (4.62) and N_1 by Theorem 4.10. Since by Lemma 4.14, $\Phi_{s,\rho} : \tilde{B}^{s+1}_\rho \to \Phi_{s,\rho}(\tilde{B}^{s+1}_\rho)$ is a (real analytic) diffeomorphism it follows that the preimage of $\Phi_{s,\rho}(\tilde{B}^{s+1}_\rho) \cap \big(\bigcup_{M \geq M_0} H^{s+1}_{\mathbb{C}} \times \mathcal{G}_{s,M}\big)$ is dense in $\tilde{B}^{s+1}_\rho$. We claim that any element in this set is a right sided finite gap potential. Indeed, if for any given $(q, \varphi) \in \tilde{B}^{s+1}_\rho$, $\Phi_{s,\rho}(q, \varphi) = (q, \mathcal{F}_{s,\rho}(q, \varphi))$ is in $H^{s+1}_{\mathbb{C}} \times \mathcal{G}_{s,M}$ for some $M \geq M_0$, then by the definition (4.62) of $\mathcal{F}_{s,\rho}$, $b^-_n(\sigma_n(q, \varphi), q, \varphi) = 0$ and $b^+_n(\sigma_n(q, \varphi), q, \varphi) = 0$ for any $n > M$. Since $(q, \varphi) \in \tilde{B}^{s+1}_\rho$ and $M \geq M_{s,\rho}$. $S_n(\lambda, q, \varphi)$ is well defined for $\lambda \in \Pi_n$ (cf. (4.41)). Since $M \geq N_1$, it follows from Theorem 4.10 that

$$\det S_n(\sigma_n(q, \varphi)) = \big(\sigma_n(q, \varphi) - \pi n - a_n(\sigma_n(q, \varphi))\big)^2 - b^+_n b^-_n\Big|_{\sigma_n(q,\varphi)} = 0.$$

Note that $\sigma_n(q, \varphi)$ is a double root of $\big(\sigma_n(q, \varphi) - \pi n - a_n(\sigma_n(q, \varphi))\big)^2$ as well as a double root of $b^+_n b^-_n$ hence it is a double root of $\det S_n$ in Π_n, implying that $\gamma_n(q, \varphi) = 0$. Hence (q, φ) is a right sided M-gap potential. We thus have shown that $\mathcal{G}^{s+1}_{\rho,\text{right}}$ is dense in $\tilde{B}^{s+1}_\rho$. Using Lemma 2.14 (reciprocity in λ) one sees that the arguments above applied to $(-q, \varphi)$ yield that $\tilde{\mathcal{G}}^{s+1}_{\rho,\text{left}}$ is also dense in $\tilde{B}^{s+1}_\rho$. This proves (i). Item (ii) is proved in the same way. □

5 Gradients

In Section 5.1 we first compute the gradient of $\grave{M}(\lambda, v) = M(1, \lambda, v)$ with respect to the L^2-pairing between $H^1_{\mathbb{C}}$ and $H^{-1}_{\mathbb{C}}$ and then use it to deduce formulas for the gradients of simple eigenvalues of $Q(v)$ such as Dirichlet or Neumann eigenvalues. In Section 5.2 we compute Floquet solutions of $Q(v)$ and use them to simplify the formulas for the gradients of eigenvalues found in Section 5.1. We denote by $\mathrm{d}F$ the differential of a map $F : H^1_c \to X$ between H^1_c and a complex Banach space X and by $\mathrm{d}F[\mathring{v}]$ the directional derivative of F in direction $\mathring{v} = (\mathring{q}, \mathring{p}) \in H^1_c$. Furthermore if F takes values in $\mathbb{C}$ then $\partial_q F, \partial_p F$ denote the L^2-gradients of F with respect to q and p and $\partial F \equiv \partial_v F$ the one of F, $\partial F = (\partial_q F, \partial_p F)$. Here $\langle \partial_q F, \mathring{q} \rangle_r = \mathrm{d}F[\mathring{q}, 0]$ where $\langle \cdot, \cdot \rangle_r$ denotes the L^2-pairing (no complex conjugation)

$$\langle f, g \rangle_r = \int_0^1 f(x) g(x) \, \mathrm{d}x \quad \forall\ f, g \in L^2(\mathbb{T}, \mathbb{C})$$

and its extension to pairings between $H^n_{\mathbb{C}}$ and $H^{-n}_{\mathbb{C}}$.

5.1 Formulas for gradients

Recall that $M(x, \lambda, v)$ denotes the fundamental solution of (2.3) and $I, J, Z, R,$ and P are given by

$$I = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \quad J = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, \quad Z = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \quad R = \begin{pmatrix} \mathrm{i} & \\ & -\mathrm{i} \end{pmatrix}, \quad P = \sqrt{1 - \partial_x^2}.$$

For a matrix valued function $x \mapsto A(x)$, $P(A)(x)$ is the matrix obtained by applying P to each matrix coefficient of A. Furthermore, we denote by EV_0 the evaluation map $H^1_c \to \mathbb{C}, (q, p) \mapsto q(0)$.

***Proposition* 5.1** *For any fixed $\lambda \in \mathbb{C}$, the $L^2_{\mathbb{C}}$ gradient of the Floquet matrix $\grave{M}(\lambda, v) = M(1, \lambda, v)$ at $v \in H^1_c$ is given by*

$$\partial_q \grave{M} = -\frac{1}{4} \grave{M} M^{-1} \mathrm{i} R M \cdot \partial_x(\cdot) - \frac{1}{16\,\lambda} \grave{M} M^{-1} \begin{pmatrix} & e^q \\ e^{-q} & \end{pmatrix} M \tag{5.1}$$

$$\partial_p \grave{M} = -\frac{\mathrm{i}}{4} \grave{M} M^{-1} R M \cdot P(\cdot). \tag{5.2}$$

Alternatively $\partial_q \grave{M}$ can be written as

$$\partial_q \grave{M} = \frac{1}{2}\begin{pmatrix} & \grave{m}_2 \\ -\grave{m}_3 & \end{pmatrix} EV_0 - \frac{1}{2}\grave{M}M^{-1}\Big(\lambda Z + \frac{1}{16\,\lambda}\begin{pmatrix} & e^q \\ e^{-q} & \end{pmatrix}\Big)M. \tag{5.3}$$

Here $\partial_q \grave{M}$ and $\partial_p \grave{M}$ are viewed as elements in $H^{-1}(\mathbb{T}, Mat_{2\times 2}(\mathbb{C}))$. Denoting by $\langle\cdot,\cdot\rangle_r$ the $L^2_{\mathbb{C}}$-pairing extended to the pairing between the Sobolev space H^1 and its dual H^{-1} it follows that for any $\mathring{p} \in H^1(\mathbb{T},\mathbb{C})$, $\langle \partial_p \grave{M}, \mathring{p}\rangle_r$ equals $\langle -\frac{\mathrm{i}}{4}\grave{M}M^{-1}RM\, ,\, P(\mathring{p})\rangle_r$ and similarly for any $\mathring{q} \in H^1(\mathbb{T},\mathbb{C})$, $\langle \partial_q \grave{M}, \mathring{q}\rangle_r$ stands for $\langle -\frac{1}{4}\grave{M}M^{-1}\mathrm{i}RM, \partial_x \mathring{q}\rangle_r - \langle \frac{1}{16\,\lambda}\grave{M}M^{-1}\begin{pmatrix} & e^q \\ e^{-q} & \end{pmatrix}M, \mathring{q}\rangle_r$.

Proof. By Theorem 2.2 the Floquet matrix $M(1,\lambda,v)$ is analytic in v. Since all terms in the above formulas depend continuously on v it suffices to verify them for sufficiently smooth v for which we may interchange differentiation with respect to x and v. For $\mathring{v} = (\mathring{q},\mathring{p}) \in H^1_c$ take the derivative of both sides of equation (2.3) in direction $\mathring{v}$, to obtain

$$\partial_x \mathrm{d}M[\mathring{v}] = J\left(\lambda - A - B^2/\lambda\right)\mathrm{d}M[\mathring{v}] - J\left(\mathrm{d}A[\mathring{v}] + \mathrm{d}(B^2)[\mathring{v}]/\lambda\right)M \tag{5.4}$$

where by (1.9)

$$\mathrm{d}A[\mathring{v}] = -\frac{1}{4}(P\mathring{p} + \mathring{q}_x)Z \qquad \mathrm{d}(B^2)[\mathring{v}] = \frac{1}{16}\begin{pmatrix} -e^{-q} & \\ & e^q \end{pmatrix}\mathring{q}.$$

Furthermore since $M(x)|_{x=0} = Id_{2\times 2}$ one has $\mathrm{d}M(x)[\mathring{v}]\Big|_{x=0} = 0$. Since $\mathrm{d}M(x)[\mathring{v}]$ solves the linear differential equation (5.4) it then can be written as

$$\mathrm{d}M(x)[\mathring{v}] = -M(x)\int_0^x M^{-1}(s)J(\mathrm{d}A[\mathring{v}] + \mathrm{d}(B^2)[\mathring{v}]/\lambda)M(s)\mathrm{d}s. \tag{5.5}$$

For $\mathring{q} = 0$ the integrand equals

$$M^{-1}(s)J\mathrm{d}A[0,\mathring{p}]M(s) = -\frac{1}{4}(P\mathring{p})M^{-1}(s)JZM(s) = \frac{1}{4}(P\mathring{p})M^{-1}(s)\mathrm{i}RM(s).$$

Evaluating $\mathrm{d}M(x)[0,\mathring{p}]$ at $x = 1$ yields the claimed formula (5.2) for $\partial_p \grave{M}$.

For $\mathring{p} = 0$, the integrand of (5.5) equals

$$\begin{aligned} &M^{-1}(s)J(\mathrm{d}A[\mathring{q},0] + \mathrm{d}B^2[\mathring{q},0]/\lambda)M(s) \\ &\qquad = \frac{1}{4}\mathring{q}_x M^{-1}\mathrm{i}RM + \mathring{q}\frac{1}{16\,\lambda}M^{-1}J\begin{pmatrix} -e^{-q} & \\ & e^q \end{pmatrix}M. \end{aligned} \tag{5.6}$$

Evaluating $-M(x)\int_0^x M^{-1}(s)J\mathrm{d}A[\mathring{q},0]M(s)\mathrm{d}s$ at $x = 1$ one obtains (5.1). Furthermore integrating by parts yields

$$-\grave{M}\int_0^1 \frac{1}{4}\mathring{q}_x(s)M^{-1}(s)\mathrm{i}RM(s)\,\mathrm{d}s = \frac{1}{2}\begin{pmatrix} & \grave{m}_2 \\ -\grave{m}_3 & \end{pmatrix}\mathring{q}(0) + \frac{1}{4}\grave{M}\int_0^1 \mathring{q}(s)\partial_s(M^{-1}(s)\mathrm{i}RM(s))\,\mathrm{d}s.$$

Since

$$\begin{aligned} \partial_s(M^{-1}\mathrm{i}RM) &= -M^{-1}(\partial_s M)M^{-1}\mathrm{i}RM + M^{-1}\mathrm{i}R(\partial_s M) = M^{-1}[\mathrm{i}R\,,\,(\partial_s M)M^{-1}]M \\ &= M^{-1}[\mathrm{i}R\,,\,J(\lambda - A - B^2/\lambda)]M = M^{-1}\Big(-2\lambda Z + \frac{2}{16\,\lambda}\begin{pmatrix} & e^q \\ e^{-q} & \end{pmatrix}\Big)M \end{aligned}$$

(where $[A,B]$ denotes the commutator $AB - BA$ of two square matrices A, B) one concludes that

$$\begin{aligned} \mathrm{d}\grave{M}[\mathring{q},0] &= -\grave{M}\int_0^1 M^{-1}(s)J\Big(\mathrm{d}A[\mathring{q},0] + \mathrm{d}B^2[\mathring{q},0]/\lambda\Big)M(s)\,\mathrm{d}x \\ &= \frac{1}{2}\begin{pmatrix} & \grave{m}_2 \\ -\grave{m}_3 & \end{pmatrix}\mathring{q}(0) \\ &\quad -\frac{1}{2}\grave{M}\int_0^1 \mathring{q}M^{-1}\Big(\lambda Z - \frac{1}{16\,\lambda}\begin{pmatrix} & e^q \\ e^{-q} & \end{pmatrix}\Big)M + \mathring{q}\frac{2}{16\,\lambda}M^{-1}J\begin{pmatrix} -e^{-q} & \\ & e^q \end{pmatrix}M\,\mathrm{d}x \end{aligned}$$

or

$$\partial_q \grave{M} = \frac{1}{2}\begin{pmatrix} & \grave{m}_2 \\ -\grave{m}_3 & \end{pmatrix}Ev_0 - \frac{1}{2}\grave{M}M^{-1}\Big(\lambda Z + \frac{1}{16\,\lambda}\begin{pmatrix} & e^q \\ e^{-q} & \end{pmatrix}\Big)M.$$

This proves (5.3) ☐

Proposition 5.1 can be used to compute the gradients of the discriminant $\Delta = (\grave{m}_1 + \grave{m}_4)/2$ and the anti-discriminant $\delta = (\grave{m}_1 - \grave{m}_4)/2$.

Lemma 5.2 (i) *For any fixed $\lambda \in \mathbb{C}^*$, the gradient of $\Delta = \Delta(\lambda)$ at $v \in H^1_c$ is given by*

$$\begin{aligned}\partial_q\Delta =&\frac{\lambda}{4}\Big(\grave{m}_2(m_3^2 - m_1^2) + \grave{m}_3(m_2^2 - m_4^2) + 2\delta\cdot(m_1m_2 - m_3m_4)\Big)\\&+\frac{1}{64\,\lambda}\Big(e^{-q}(\grave{m}_3m_2^2 - \grave{m}_2m_1^2 + 2\delta\cdot m_1m_2) + e^{q}(\grave{m}_2m_3^2 - \grave{m}_3m_4^2 - 2\delta\cdot m_3m_4)\Big)\\\partial_p\Delta =&\frac{1}{4}\Big(-\grave{m}_2m_1m_3 + \grave{m}_3m_2m_4 + \delta\cdot(m_1m_4 + m_2m_3)\Big)P(\cdot),\end{aligned}$$

(ii) *For any fixed $\lambda \in \mathbb{C}$, the gradient of the anti-discriminant is given by*

$$\begin{aligned}\partial_q\delta =&\frac{\lambda}{4}\Big(\grave{m}_2(m_3^2 - m_1^2) + \grave{m}_3(m_4^2 - m_2^2) + 2\Delta\cdot(m_1m_2 - m_3m_4)\Big)\\&-\frac{1}{64\,\lambda}\Big(e^{-q}(\grave{m}_2m_1^2 + \grave{m}_3m_2^2 - 2\Delta\cdot m_1m_2) + e^{q}(-\grave{m}_3m_4^2 - \grave{m}_2m_3^2 + 2\Delta\cdot m_3m_4)\Big)\\\partial_p\delta =&\frac{1}{4}\Big(-\grave{m}_3m_2m_4 - \grave{m}_2m_1m_3 + \Delta\cdot(m_1m_4 + m_2m_3)\Big)P(\cdot).\end{aligned}$$

(iii) *At the zero potential $v = 0$, one has $\partial\Delta = 0$ and $\partial\delta$ is given by*

$$\begin{aligned}\partial_q\delta(\lambda,0) =&\frac{1}{2}\Big(\lambda + \frac{1}{16\,\lambda}\Big)\Big(\cos(\omega(\lambda))\sin(2\omega(\lambda)x) - \sin(\omega(\lambda))\cos(2\omega(\lambda)x)\Big)\\\partial_p\delta(\lambda,0) =&\frac{\cos(\omega(\lambda))}{4}\Big(\cos(2\omega(\lambda)x)\Big)P(\cdot) + \frac{\sin(\omega(\lambda))}{4}\Big(\sin(2\omega(\lambda)x)\Big)P(\cdot).\end{aligned}$$

Proof. Items (i) and (ii) follow from Proposition 5.1. To prove item (iii) substitute $E_{\omega(\lambda)}$, defined in (2.31), for M into the formulas in item (i) to conclude that $\partial\Delta$ vanishes. For δ we obtain

$$\partial_q\delta(\lambda,0) =\frac{1}{4}\Big(\lambda + \frac{1}{16\,\lambda}\Big)\Big(2\sin(\omega(\lambda))\big(\sin^2(\omega(\lambda)x) - \cos^2(\omega(\lambda)x)\big) + 4\cos(\omega(\lambda))\sin(\omega(\lambda)x)\cos(\omega(\lambda)x)\Big)$$

and

$$\partial_p\delta(\lambda,0) =\frac{\cos(\omega(\lambda))}{4}\Big(\cos^2(\omega(\lambda)x) - \sin^2(\omega(\lambda)x)\Big)P(\cdot) + \frac{1}{2}\sin(\omega(\lambda))\Big(\sin(\omega(\lambda)x)\cos(\omega(\lambda)x)\Big)P(\cdot)$$

which can be further simplified using that $\cos^2(x) - \sin^2(x) = \cos(2x)$ and $2\sin(x)\cos(x) = \sin(2x)$. □

Next we want to obtain formulas for the gradient of simple periodic, Dirichlet, and Neumann eigenvalues. First we prove the following auxiliary result.

Recall that a solution of $QF = \lambda F$ has the form $F = \begin{pmatrix} f \\ \frac{1}{\lambda}Bf\end{pmatrix}$ with $f(x) = M(x,\lambda)a$ and $a \in \mathbb{C}^2$.

Proposition 5.3 *Assume that $\kappa \equiv \kappa(v) \in \mathbb{C}^*$ and $a(v) \in \mathbb{C}^2$ are analytic functions on some open set in H^1_c and define $f(x,v) = (f_1(x,v), f_2(x,v)) = M(x,\kappa(v),v)a(v) \in H^1([0,1],\mathbb{C}^2)$. Then for any $\mathring{v} = (\mathring{q},\mathring{p}) \in H^1_c$, the derivative $\mathrm{d}\kappa[\mathring{v}]$ of κ at v in direction $\mathring{v}$ is given by*

$$\begin{aligned}\mathrm{d}\kappa[\mathring{v}] =&\frac{1}{\int_0^1 f\cdot(I + \frac{1}{\kappa^2}B^2)f\,\mathrm{d}x}\Big(\Big[\mathrm{d}f[\mathring{v}]\cdot Jf - \frac{1}{2}\mathring{q}f_1f_2\Big]_0^1 + \int_0^1\Big(\frac{\kappa}{2}(f_2^2 - f_1^2) + \frac{1}{32\,\kappa}(f_2^2e^{q} - f_1^2e^{-q})\Big)\mathring{q}\,\mathrm{d}x\\&-\frac{1}{2}\int_0^1 f_1f_2P(\mathring{p})\,\mathrm{d}x\Big).\end{aligned}$$

To prove Proposition 5.3 we first need to derive the following two lemmas.

Lemma 5.4 *Let $\kappa(v)$ and $f(v) = (f_1(v), f_2(v))$ be given as in Proposition 5.3. Then for any $\mathring{v} = (\mathring{q},\mathring{p}) \in H^1_c$,*

$$\mathrm{d}\kappa[\mathring{v}] = \frac{1}{\int_0^1 f\cdot(I + \frac{1}{\kappa^2}B^2)f\,\mathrm{d}x}\left([\mathrm{d}f[\mathring{v}]\cdot Jf]_0^1 + \int_0^1\frac{1}{16\kappa}(f_2^2e^{q} - f_1^2e^{-q})\mathring{q} - \frac{1}{2}(P(\mathring{p}) + \partial_x\mathring{q})f_1f_2\,\mathrm{d}x\right). \tag{5.7}$$

Proof of Lemma 5.4. Let $F = \begin{pmatrix} f \\ \frac{1}{\kappa}Bf \end{pmatrix}$ and differentiate both sides of

$$QF(x,v) = \kappa(v)F(x,v)$$

in the direction $\mathring{v} = (\mathring{q},\mathring{p}) \in H^1_c$ to obtain

$$(\mathrm{d}Q[\mathring{v}])F + Q\,\mathrm{d}F[\mathring{v}] = (\mathrm{d}\kappa[\mathring{v}])F + \kappa\,\mathrm{d}F[\mathring{v}]. \tag{5.8}$$

Take the L^2 pairing product with F. Since $Q_0(v)$ is symmetric one has

$$\begin{aligned}\int_0^1 (Q\,\mathrm{d}F[\mathring{v}] - \kappa\,\mathrm{d}F[\mathring{v}])\cdot F\,\mathrm{d}x &= \int_0^1 (Q_1\partial_x + Q_0)\,\mathrm{d}F[\mathring{v}]\cdot F - \mathrm{d}F[\mathring{v}]\cdot(Q_1\partial_x + Q_0)F \\ &= \int_0^1 \left(Q_1\partial_x\,\mathrm{d}F[\mathring{v}]\right)\cdot F - \mathrm{d}F[\mathring{v}]\cdot(Q_1\partial_x F) \\ &= \int_0^1 -J\partial_x(\mathrm{d}f[\mathring{v}])\cdot f - \mathrm{d}f[\mathring{v}]\cdot(-J\partial_x f)\,\mathrm{d}x = [\mathrm{d}f[\mathring{v}]\cdot Jf]_0^1.\end{aligned}$$

Thus (5.8) yields

$$\int_0^1 (\mathrm{d}Q[\mathring{v}])F\cdot F\,\mathrm{d}x + [\mathrm{d}f[\mathring{v}]\cdot Jf]_0^1 = \mathrm{d}\kappa[\mathring{v}]\int_0^1 F\cdot F\,\mathrm{d}x = \mathrm{d}\kappa[\mathring{v}]\int_0^1 f(I + \frac{1}{\kappa^2}B^2)f\,\mathrm{d}x. \tag{5.9}$$

Finally

$$\begin{aligned}(\mathrm{d}Q[\mathring{v}]F)\cdot F &= \begin{pmatrix} \mathrm{d}A[\mathring{v}] & \mathrm{d}B[\mathring{v}] \\ \mathrm{d}B[\mathring{v}] & \end{pmatrix}\begin{pmatrix} f \\ \frac{1}{\kappa}Bf \end{pmatrix}\cdot\begin{pmatrix} f \\ \frac{1}{\kappa}Bf \end{pmatrix} \\ &= \mathrm{d}A[\mathring{v}]f\cdot f + \frac{1}{\kappa}(\mathrm{d}B[\mathring{v}])Bf\cdot f + \frac{1}{\kappa}(\mathrm{d}B[\mathring{v}])f\cdot Bf \\ &= \mathrm{d}A[\mathring{v}]f\cdot f + \frac{2}{\kappa}(\mathrm{d}B[\mathring{v}])Bf\cdot f\end{aligned}$$

where $\frac{2}{\kappa}(\mathrm{d}B[\mathring{v}])B = \frac{1}{16\,\kappa}\begin{pmatrix} -e^{-q} & \\ & e^q \end{pmatrix}\mathring{q}$ and

$$\mathrm{d}A[\mathring{v}]f\cdot f = -\frac{1}{4}(P\mathring{p} + \partial_x\mathring{q})f\cdot Zf.$$

Hence

$$\mathrm{d}Q[\mathring{v}]F\cdot F = -\frac{1}{2}(P(\mathring{p}) + \partial_x\mathring{q})f_1f_2 + \frac{1}{16\kappa}\left(f_2^2e^q - f_1^2e^{-q}\right)\mathring{q}.$$

Combining this with (5.9) leads to the claimed identity. □

Lemma 5.5 *For $v \in H^1_c$, $\lambda,\mu \in \mathbb{C}^*$ and $a_\lambda, a_\mu \in \mathbb{C}^2$, let $f(x) = (f_1(x), f_2(x)) = M(x,\lambda,v)a_\lambda$ and $g(x) = (g_1(x), g_2(x)) = M(x,\mu,v)a_\mu$. Then*

$$\partial_x(f_1g_2 + f_2g_1) = (\lambda+\mu)(f_2g_2 - f_1g_1) + \frac{1}{16}\left(\frac{1}{\lambda} + \frac{1}{\mu}\right)(f_1g_1e^{-q} - f_2g_2e^q).$$

Proof of Lemma 5.5. Using that f and g fulfill (2.3) we compute

$$\begin{aligned}\partial_x(f_1g_1 + f_2g_2) &= \partial_x(f\cdot Zg) = (J(\lambda - A - B^2/\lambda)f\cdot Zg) + (f\cdot ZJ(\mu - A - B^2/\mu)g) \\ &= (\lambda+\mu)f\cdot ZJg + \frac{1}{4}(P(p) + q_x)\left(JZf\cdot Zg + f\cdot ZJZg\right) \\ &\quad - \frac{1}{16}\left(\frac{1}{\lambda}Je^{\mathrm{i}Rq}f\cdot Zg + \frac{1}{\mu}f\cdot ZJe^{\mathrm{i}Rq}g\right) \\ &= (\lambda+\mu)(f_2g_2 - f_1g_1) + \frac{1}{16}\left(\frac{1}{\lambda} + \frac{1}{\mu}\right)(f_1g_1e^{-q} - f_2g_2e^q).\end{aligned}$$

□

Proof of Proposition 5.3. Our starting point is formula (5.7). To obtain a formula for the directional derivative $d\kappa[\mathring{v}]$ we need to integrate the term $-\frac{1}{2}\int_0^1(\partial_x\mathring{q})f_1f_2\,dx$ in (5.7) by parts. Since by Lemma 5.5 with $g = f$ and $\lambda = \kappa$, $\mu = \kappa$

$$\int_0^1(\partial_x\mathring{q})f_1f_2\,dx = [\mathring{q}f_1f_2]_0^1 - \int_0^1 \kappa\mathring{q}(f_2^2 - f_1^2) + \frac{1}{16\,\kappa}\mathring{q}(f_1^2e^{-q} - f_2^2e^q)\,dx$$

formula (5.7) yields

$$\begin{aligned} d\kappa[\mathring{v}] =& \frac{1}{\int_0^1 f\cdot(I+\frac{1}{\kappa^2}B^2)f\,dx}\Big(\Big[df[\mathring{v}]\cdot Jf - \frac{1}{2}\mathring{q}f_1f_2\Big]_0^1 + \int_0^1\Big(\frac{\kappa}{2}(f_2^2 - f_1^2) + \frac{1}{32\,\kappa}(f_2^2e^q - f_1^2e^{-q})\Big)\mathring{q}\,dx \\ &-\frac{1}{2}\int_0^1 f_1f_2P(\mathring{p})\,dx\Big). \end{aligned}$$

□

For what follows it is useful to denote the first and second columns of M by M_1 and M_2 respectively. By a slight abuse of terminology we write M_1 and M_2 as row vectors (m_1, m_3) and respectively, (m_2, m_4). Assume that on an open set V in H_c^1, $\lambda = \lambda(v)$ is a simple periodic, Neumann, or Dirichlet eigenvalue. Then $\lambda(v)$ is analytic in v and we can choose a corresponding eigenfunction $f = (f_1, f_2) \in H^1([0,1], \mathbb{C}^2)$ which is analytic as a function of v on V. It is straightforward to verify that

$$df[\mathring{v}]\cdot Jf\Big|_0^1 = \frac{1}{2}f_1f_2\mathring{q}\Big|_0^1 = 0.$$

Introduce for $f = (f_1, f_2) \in L_c^2$

$$[\![f]\!]_{q,\lambda} := \begin{pmatrix} \frac{\lambda}{2}(f_2^2 - f_1^2) + \frac{1}{32\lambda}(f_2^2e^q - f_1^2e^{-q}) \\ -\frac{1}{2}f_1f_2P(\cdot) \end{pmatrix} \tag{5.10}$$

In case $f \equiv f_\lambda$ depends on the spectral parameter λ we will write often $[\![f]\!]_{q,\lambda}$ instead of $[\![f_\lambda]\!]_{q,\lambda}$. Then according to Proposition 5.3

$$d\lambda[\mathring{v}] = \frac{1}{\int_0^1 f\cdot(I+\frac{1}{\lambda^2}B^2)f\,dx}\int_0^1 [\![f]\!]_{q,\lambda}\cdot\mathring{v}\,dx, \tag{5.11}$$

Lemma 5.6 *Assume $\kappa_0 \in C^*$ is a simple Dirichlet eigenvalue of $Q(v_0)$ for $v_0 \in H_c^1$. Then there exists a neighborhood V of v_0 in H_c^1 and $\epsilon > 0$ such that there is a unique Dirichlet eigenvalue $\kappa(v)$ of $Q(v)$ in the disc of radius ϵ centered at κ_0 for all $v \in V$. κ is analytic on V and $\partial\kappa = \frac{\grave{m}_1}{\dot{\chi}_D}[\![M_2]\!]_{q,\kappa}$ where $M_2 = M_2(\cdot,\kappa(v))$ is the canonical eigenfunction for $\kappa(v)$. More explicitly*

$$d\kappa[\mathring{v}] = -\frac{d\grave{m}_2[\mathring{v}]\big|_{\lambda=\kappa}}{\dot{\chi}_D(\kappa)} = \frac{\grave{m}_1(\kappa)}{\dot{\chi}_D(\kappa)}\int_0^1\begin{pmatrix}\frac{\kappa}{2}(m_4^2 - m_2^2) + \frac{1}{32\kappa}(m_4^2e^q - m_2^2e^{-q}) \\ -\frac{1}{2}m_2m_4\end{pmatrix}\cdot\begin{pmatrix}\mathring{q} \\ P(\mathring{p})\end{pmatrix}dx.$$

Proof. By the argument principle existence and analyticity of κ follow. By (5.10)-(5.11) we have

$$d\kappa[0,\mathring{p}] = \frac{1}{\int_0^1 M_2\cdot(I+\frac{1}{\kappa^2}B^2)M_2\,dx}\int_0^1 -\frac{1}{2}m_2m_4P(\mathring{p})\,dx. \tag{5.12}$$

On the other hand, applying the chain rule to $\chi_D(\kappa(v), v) = 0$ and using that $\dot{\chi}_D(\kappa) \neq 0$ since κ is simple we get

$$d\kappa[\mathring{v}] = -\frac{d\chi_D[\mathring{v}]\big|_{\lambda=\kappa}}{\dot{\chi}_D(\kappa)} = -\frac{d\grave{m}_2[\mathring{v}]\big|_{\lambda=\kappa}}{\dot{\chi}_D(\kappa)} \tag{5.13}$$

where by Proposition 5.1

$$d\grave{m}_2(\kappa)[0,\mathring{p}] = \frac{1}{4}\int_0^1(2\grave{m}_1m_2m_4 - \grave{m}_2(m_1m_4 + m_2m_3))\Big|_{\lambda=\kappa}P(\mathring{p})\,dx.$$

Since at a Dirichlet eigenvalue, $\grave{m}_2 = 0$, one obtains by comparing (5.12) and (5.13),

$$\frac{1}{\int_0^1 M_2\cdot(I+\frac{1}{\kappa^2}B^2)M_2\,dx} = \frac{\grave{m}_1(\kappa)}{\dot{\chi}_D(\kappa)}.$$

This together with (5.11) yields the claimed formula. □

***Lemma* 5.7** *Assume $\kappa_0 \in C^*$ is a simple Neumann eigenvalue of $Q(v_0)$ for $v_0 \in H^1_c$. Then there exists a neighborhood V of v_0 in H^1_c and $\epsilon > 0$ such that for any $v \in V$ there is a unique Neumann eigenvalue $\kappa(v)$ of $Q(v)$ in the disc of radius ϵ centered at κ_0. κ is analytic on V and $\partial\kappa = -\frac{\grave{m}_4}{\dot{\chi}_N} [\![M_1]\!]_{q,\kappa}$, where $M_1 \equiv M_1(\cdot,\kappa)$ is the canonical eigenfunction for $\kappa(q,p)$. More explicitly,*

$$\mathrm{d}\kappa[\mathring{v}] = -\frac{\mathrm{d}\grave{m}_3[\mathring{v}]\big|_{\lambda=\kappa}}{\dot{\chi}_N(\kappa)} = -\frac{\grave{m}_4(\kappa)}{\dot{\chi}_N(\kappa)} \int_0^1 \begin{pmatrix} \frac{\kappa}{2}(m_3^2 - m_1^2) + \frac{1}{32\kappa}(m_3^2 e^q - m_1^2 e^{-q}) \\ -\frac{1}{2} m_1 m_3 \end{pmatrix} \cdot \begin{pmatrix} \mathring{q} \\ P(\mathring{p}) \end{pmatrix} \mathrm{d}x.$$

Proof. By the argument principle existence and analyticity of κ follow. By (5.10)-(5.11) one has

$$\mathrm{d}\kappa[0,\mathring{p}] = \frac{1}{\int_0^1 M_1 \cdot (1 + \frac{1}{\kappa^2} B^2) M_1} \int_0^1 -\frac{1}{2} m_1 m_3 P(\mathring{p})\,\mathrm{d}x$$

On the other hand, applying the chain rule to $\chi_N(\kappa(q,p),q,p) = 0$ one gets

$$\mathrm{d}\kappa[\mathring{v}] = -\frac{\mathrm{d}\chi_N[\mathring{v}]\big|_{\lambda=\kappa}}{\dot{\chi}_N(\kappa)} = -\frac{\mathrm{d}\grave{m}_3[\mathring{v}]\big|_{\lambda=\kappa}}{\dot{\chi}_N(\kappa)}$$

where by Proposition 5.1

$$\mathrm{d}\grave{m}_3(\kappa)[0,\mathring{p}] = -\frac{1}{4} \int_0^1 (2\grave{m}_4 m_1 m_3 - \grave{m}_3(m_1 m_4 + m_2 m_3))\Big|_{\lambda=\kappa} P(\mathring{p})\,\mathrm{d}x.$$

Since κ is a Neumann eigenvalue $\grave{m}_3(\kappa) = 0$ and one obtains

$$\frac{1}{\int_0^1 M_1 \cdot (I + \frac{1}{\kappa^2} B^2) M_1\,\mathrm{d}x} = -\frac{\grave{m}_4(\kappa)}{\dot{\chi}_N(\kappa)}.$$

□

5.2 Floquet solutions

Fix $v \in H^1_c$, $\lambda \in \mathbb{C}^*$ and consider the two eigenvalues $\xi_\pm = \Delta(\lambda) \pm \sqrt{\Delta^2(\lambda) - 1}$ of $\grave{M} = \grave{M}(\lambda, v)$. If λ is a periodic eigenvalue then $\xi_+ = \xi_- \in \{1, -1\}$. For any other λ, $\xi_+ \neq \xi_-$ and there exist corresponding eigenvectors $a_+, a_- \in \mathbb{C}^2$. Hence, except when λ is a simple periodic eigenvalue, there are two distinct Floquet solutions, $f_+ \equiv (f_1^+, f_2^+) := M(x,\lambda,v)a_+$ and $f_- \equiv (f_1^-, f_2^-) := M(x,\lambda,v)a_-$. Using these two solutions, one can rewrite the gradient of M and hence of Δ and δ. Since a_+, a_- are linearly independent the 2×2 matrix $(a_+\ a_-)$ with columns a_+ and a_- is regular. Note that

$$(f_+\ f_-)\,(a_+\ a_-)^{-1} = M\,(a_+\ a_-)\,(a_+\ a_-)^{-1} = M. \tag{5.14}$$

This suggests that the gradient of $\grave{M}$ can be written in terms of the Floquet solutions. To simplify the formulas below, let $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}^\perp = (-a_2, a_1)$ for any vector $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \in \mathbb{C}^2$. Then for any linearly independent vectors $a, b \in \mathbb{C}^2$ the determinant of the 2×2 matrix $(a \quad b)$ is the 1×1 matrix $a^\perp b$ and the matrix inverse $(a \quad b)^{-1}$ can be written as

$$(a\ b)^{-1} = \frac{1}{a^\perp b} \begin{pmatrix} -b^\perp \\ a^\perp \end{pmatrix}. \tag{5.15}$$

Furthermore we introduce the star product

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \star \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = a_1 b_1 - a_2 b_2.$$

***Lemma* 5.8** *Let $v \in H^1_c$ and assume that $\lambda \in \mathbb{C}^*$ is not a simple periodic eigenvalue of $Q(v)$. Then,*

$$\begin{aligned} \partial_q \grave{M} =& \frac{1}{2} \begin{pmatrix} & \grave{m}_2 \\ -\grave{m}_3 & \end{pmatrix} EV_0 + \frac{\lambda}{2a_+^\perp a_-} \begin{pmatrix} \xi_+ a_+ & \xi_- a_- \end{pmatrix} \begin{pmatrix} f_+ \star f_- & f_- \star f_- \\ -f_+ \star f_+ & -f_+ \star f_- \end{pmatrix} (a_+\ a_-)^{-1} \\ &+ \frac{1}{32\,\lambda\, a_+^\perp a_-} \begin{pmatrix} \xi_+ a_+ & \xi_- a_- \end{pmatrix} \left(e^q \begin{pmatrix} -f_2^+ f_2^- & -(f_2^-)^2 \\ (f_2^+)^2 & f_2^+ f_2^- \end{pmatrix} + e^{-q} \begin{pmatrix} f_1^+ f_1^- & (f_1^-)^2 \\ -(f_1^+)^2 & -f_1^+ f_1^- \end{pmatrix} \right) (a_+\ a_-)^{-1} \\ \partial_p \grave{M} =& \frac{1}{4a_+^\perp a_-} \begin{pmatrix} \xi_+ a_+ & \xi_- a_- \end{pmatrix} \begin{pmatrix} f_+ \cdot Z f_- & f_- \cdot Z f_- \\ -f_+ \cdot Z f_+ & -f_+ \cdot Z f_- \end{pmatrix} \begin{pmatrix} a_+ & a_- \end{pmatrix}^{-1} P(\cdot). \end{aligned}$$

Proof. By Proposition 5.1 and (5.14)

$$\begin{aligned}\partial_p \grave{M} &= -\frac{\mathrm{i}}{4}\grave{M}M^{-1}RMP(\cdot)\\ &= -\frac{\mathrm{i}}{4}\begin{pmatrix}\xi_+ a_+ & \xi_- a_-\end{pmatrix}(a_+\ a_-)^{-1}(a_+\ a_-)(f_+\ f_-)^{-1}R(f_+\ f_-)(a_+\ a_-)^{-1}P(\cdot).\end{aligned}$$

By (5.15) it then follows

$$\begin{aligned}\partial_p \grave{M} &= -\frac{\mathrm{i}}{4}\begin{pmatrix}\xi_+ a_+ & \xi_- a_-\end{pmatrix}\frac{1}{f_+^\perp f_-}\begin{pmatrix}-f_-^\perp\\ f_+^\perp\end{pmatrix}R(f_+\ f_-)(a_+\ a_-)^{-1}P(\cdot)\\ &=\frac{1}{4}\begin{pmatrix}\xi_+ a_+ & \xi_- a_-\end{pmatrix}\frac{1}{f_+^\perp f_-}\begin{pmatrix}f_+\cdot Zf_- & f_-\cdot Zf_-\\ -f_+\cdot Zf_+ & -f_+\cdot Zf_-\end{pmatrix}(a_+\ a_-)^{-1}P(\cdot).\end{aligned}$$

Since the 2×2 matrix $(f_+\ f_-)$ fulfills (2.3) $det(f_+\ f_-) = f_+^\perp f_-$ is constant (Wronskian identity). Hence $f_+^\perp f_- = a_+^\perp a_-$ and the claimed formula for $\partial_p\grave{M}$ is proved. Similarly, by Proposition 5.1

$$\partial_q \grave{M} = \frac{1}{2}\begin{pmatrix} & \grave{m}_2\\ -\grave{m}_3 & \end{pmatrix}EV_0 - \frac{1}{2}\grave{M}M^{-1}\Big(\lambda Z + \frac{1}{16\,\lambda}\begin{pmatrix} & e^q\\ e^{-q} & \end{pmatrix}\Big)M.$$

One computes

$$\begin{aligned}\grave{M}M^{-1}ZM &= \begin{pmatrix}\xi_+ a_+ & \xi_- a_-\end{pmatrix}(f_+\ f_-)^{-1}Z(f_+\ f_-)(a_+\ a_-)^{-1}\\ &= \begin{pmatrix}\xi_+ a_+ & \xi_- a_-\end{pmatrix}\frac{1}{f_+^\perp f_-}\begin{pmatrix}-f_+\star f_- & -f_-\star f_-\\ f_+\star f_+ & f_+\star f_-\end{pmatrix}(a_+\ a_-)^{-1}\end{aligned}$$

and

$$\begin{aligned}\grave{M}M^{-1}\begin{pmatrix} & e^q\\ e^{-q} & \end{pmatrix}M &= \begin{pmatrix}\xi_+ a_+ & \xi_- a_-\end{pmatrix}(f_+\ f_-)^{-1}\begin{pmatrix} & e^q\\ e^{-q} & \end{pmatrix}(f_+\ f_-)(a_+\ a_-)^{-1}\\ &= \begin{pmatrix}\xi_+ a_+ & \xi_- a_-\end{pmatrix}\frac{1}{f_+^\perp f_-}\left(e^q\begin{pmatrix}f_2^+f_2^- & (f_2^-)^2\\ -(f_2^+)^2 & -f_2^+f_2^-\end{pmatrix} + e^{-q}\begin{pmatrix}-f_1^+f_1^- & -(f_1^-)^2\\ (f_1^+)^2 & f_1^+f_1^-\end{pmatrix}\right)(a_+\ a_-)^{-1}.\end{aligned}$$

Using again that $f_+^\perp f_-$ is constant, the claimed formula for $\partial_q\grave{M}$ follows. □

Lemma 5.8 leads to the following formula for the gradient of $\Delta(\lambda, v)$ with respect to $v = (q, p)$.

Lemma 5.9 *Let $v\in H_c^2$ and assume that $\lambda\in\mathbb{C}^*$ is not a periodic eigenvalue of Q. Then the gradient $\partial_v\Delta$ of $\Delta = \Delta(\lambda)$ is given by*

$$\begin{aligned}\partial_q\Delta &= \frac{\xi_+ - \xi_-}{4a_+^\perp a_-}\left(\lambda f_+\star f_- + \frac{1}{16\,\lambda}\left(e^{-q}f_1^+f_1^- - e^q f_2^+f_2^-\right)\right)\\ \partial_p\Delta &= \frac{\xi_+ - \xi_-}{8a_+^\perp a_-}f_+\cdot Zf_-P(\cdot).\end{aligned}$$

The following formulas for the Floquet solutions f_+, f_- turn out to be useful to rewrite the formulas for $\partial_v\Delta$ of Lemma 5.9 in a convenient way.

Lemma 5.10 *Assume that $\lambda\in\mathbb{C}$ is not a periodic eigenvalue of Q. If $\grave{m}_2(\lambda)\neq 0$ (meaning that λ is not a Dirichlet eigenvalue), eigenvectors a_+, a_- of $\grave{M}(\lambda)$, corresponding to the eigenvalues ξ_+, ξ_- are given by*

$$a_\pm := \begin{pmatrix}\grave{m}_2\\ \xi_\pm - \grave{m}_1\end{pmatrix} \tag{5.16}$$

and (3.14) *yields the identity $\xi_\pm - \grave{m}_1 = -\delta\pm\sqrt{\Delta^2-1}$ implying that in view of* (5.16)

$$f_\pm = Ma_\pm = \grave{m}_2 M_1 - \delta\cdot M_2 \pm \sqrt{\Delta^2-1}M_2.$$

If $\grave{m}_3(\lambda)\neq 0$ (meaning that λ is not a Neumann eigenvalue), eigenvectors a_+, a_-, corresponding to the eigenvalues ξ_+, ξ_- are given by

$$a_\pm := \begin{pmatrix}\xi_\pm - \grave{m}_4\\ \grave{m}_3\end{pmatrix} \tag{5.17}$$

and (3.14) *yields the identity $\xi_\pm - \grave{m}_4 = \delta\pm\sqrt{\Delta^2-1}$, leading together with* (5.17) *to*

$$f_\pm = Ma_\pm = \grave{m}_3 M_2 + \delta\cdot M_1 \pm \sqrt{\Delta^2-1}M_1.$$

Remark 5.11. Note that the Dirichlet spectrum and Neumann spectrum of $Q(v)$ are discrete and a rough localization is provided by the corresponding Counting Lemmas – (see Lemma 3.4 and Theorem 3.6)

Taking into account that $[\![f]\!]_{q,\lambda}$, defined in (5.10), can be written in the form

$$[\![f]\!]_{q,\lambda} = -\begin{pmatrix} \frac{\lambda}{2} f \star f + \frac{1}{32\,\lambda} f \cdot \begin{pmatrix} e^{-q} & \\ & -e^{q} \end{pmatrix} f \\ \frac{1}{4} f \cdot ZfP(\cdot) \end{pmatrix} \tag{5.18}$$

Lemma 5.9 and Lemma 5.10 then lead to the following result.

***Proposition* 5.12** *Let $v \in H_c^1$. For $\lambda \in \mathbb{C}^*$ with $\grave{m}_2(\lambda) \neq 0$,*

$$\partial\Delta = \frac{1}{2\,\grave{m}_2} [\![\grave{m}_2 M_1 - \delta \cdot M_2]\!]_{q,\lambda} - \frac{\Delta^2 - 1}{2\,\grave{m}_2} [\![M_2]\!]_{q,\lambda}$$

whereas for $\lambda \in \mathbb{C}^*$ with $\grave{m}_3(\lambda) \neq 0$,

$$\partial\Delta = -\frac{1}{2\,\grave{m}_3} [\![\grave{m}_3 M_2 + \delta \cdot M_1]\!]_{q,\lambda} + \frac{\Delta^2 - 1}{2\,\grave{m}_3} [\![M_1]\!]_{q,\lambda}\,.$$

Proof. Assume that $\lambda \in \mathbb{C}^*$ is not a periodic eigenvalue. If $\grave{m}_2(\lambda) \neq 0$, (5.16) yields $a_+^{\perp} a_- = (\xi_- - \xi_+)\grave{m}_2$ whereas if $\grave{m}_3(\lambda) \neq 0$, (5.17) yields $a_+^{\perp} a_- = (\xi_+ - \xi_-)\grave{m}_3$. In the case $\lambda \in \mathbb{C}^*$ is a periodic eigenvalue such that $\grave{m}_2(\lambda) \neq 0$ or $\grave{m}_3(\lambda) \neq 0$ then $\grave{m}_2$, respectively, $\grave{m}_3$ do not vanish in a neighborhood U of λ. Since periodic eigenvalues are isolated, the corresponding claimed identities hold in $U \setminus \{\lambda\}$ and hence by continuity also at λ. □

***Lemma* 5.13** *Let $v_0 \in H_c^1$ and assume that $\kappa_0 \in \mathbb{C}^*$ is a simple periodic eigenvalue of $Q(v_0)$. Then in an open neighborhood of v_0 there exists an analytic function $\kappa = \kappa(v)$ of simple periodic eigenvalues of $Q(v)$ which coincides at v_0 with κ_0, $\dot{\Delta}(\kappa) \neq 0$, and*

$$\partial\kappa = -\frac{1}{\dot{\Delta}(\kappa)} \partial\Delta\big|_{\kappa}.$$

Furthermore, either $\grave{m}_2(\kappa, v) \neq 0$ or $\grave{m}_3(\kappa, v) \neq 0$. If $\grave{m}_2(\kappa, v) \neq 0$ then $\grave{m}_2 M_1 - \delta \cdot M_2$ is an eigenfunction for κ and

$$\partial\kappa = -\frac{1}{2\dot{\Delta}(\kappa)\grave{m}_2(\kappa)} [\![\grave{m}_2 M_1 - \delta \cdot M_2]\!]_{q,\kappa}\,.$$

If $\grave{m}_3(\kappa, v) \neq 0$ then $\grave{m}_3 M_2 + \delta \cdot M_1$ is a periodic eigenfunction for κ and

$$\partial\kappa = \frac{1}{2\dot{\Delta}(\kappa)\grave{m}_3(\kappa)} [\![\grave{m}_3 M_2 - \delta \cdot M_1]\!]_{q,\kappa}\,.$$

Proof. If κ_0 is a periodic eigenvalue of $Q(v_0)$, then $\grave{M}(\kappa_0)$ has the eigenvalue 1 or -1 and since $\det \grave{M} = 1$, it has algebraic multiplicity two. If both $\grave{m}_2(\kappa_0, v_0)$ and $\grave{m}_3(\kappa_0, v_0)$ were to vanish then $\grave{M}(\kappa_0) = \sigma I$, $\sigma \in \{1, -1\}$, and hence κ_0 would be a periodic eigenvalue of geometric multiplicity two which contradicts our assumption. Given that κ_0 is a simple periodic eigenvalue of $Q(v_0)$ it is well known that there exists an analytic function $\kappa = \kappa(v)$ on a neighborhood of v_0 such that $\kappa(v)$ is a simple eigenvalue of $Q(v)$ with $\Delta(\kappa(v), v) = \sigma$. Applying the chain rule to $\Delta(\kappa(v), v) = \sigma$, one gets

$$\partial\kappa = -\frac{1}{\dot{\Delta}(\kappa)}\, \partial\Delta\big|_{\kappa}\,.$$

The claimed formulas for $\partial\kappa$ then follow from Proposition 5.12. □

Finally, based on the formulas for the gradient of the Floquet matrix of Lemma 5.8 and the formulas of the Floquet solutions of Lemma 5.10 we provide a formula for $\partial_v \grave{m}_4(\lambda)$ at a Dirichlet eigenvalue which turns out to be useful.

***Lemma* 5.14** *Let μ be a simple Dirichlet eigenvalue of $Q(v)$, depending analytically on v in some open set V of H_c^1. Then*

$$\partial_v \grave{m}_4\big|_{\lambda=\mu} = -\grave{m}_3(\mu)\, [\![M_2]\!]_{q,\mu} + \frac{\grave{m}_4(\mu)}{4}\Big([\![M_1 + M_2]\!]_{q,\mu} - [\![M_1 - M_2]\!]_{q,\mu} \Big).$$

Proof. First let us consider the case where $\grave{m}_3(\mu) \neq 0$ and $\delta(\mu) = (\grave{m}_1(\mu) - \grave{m}_4(\mu))/2 \neq 0$ in some open subset of V. It then follows that μ is neither a periodic nor a Neumann eigenvalue. Since $\grave{m}_2(\mu) = 0$, $\grave{M}(\mu)$ is lower triangular and the two Floquet multipliers are $\xi_+ = \grave{m}_1(\mu)$ and $\xi_- = \grave{m}_4(\mu)$. (Here we have chosen the square root $\sqrt{\Delta^2(\mu) - 1}$ to be given by $\delta(\mu)$.) Corresponding eigenvectors are then given by

$$a_+ = (2\delta(\mu), \grave{m}_3(\mu)), \quad a_- = (0, \grave{m}_3(\mu)). \tag{5.19}$$

Since $\xi_+ - \xi_- = 2\delta(\mu) \neq 0$ by assumption, a_+ and a_- are linearly independent and with $a_+^\perp = (-a_2^+, a_1^+) = (-\grave{m}_3, 2\delta)$ one gets

$$a_+^\perp a_- = (-\grave{m}_3, 2\delta) \cdot (0, \grave{m}_3) = 2\delta \grave{m}_3 \tag{5.20}$$

and the inverse of $\begin{pmatrix} a_+ & a_- \end{pmatrix} = \begin{pmatrix} 2\delta & 0 \\ \grave{m}_3 & \grave{m}_3 \end{pmatrix} \in \mathbb{C}^{2\times 2}$ is given by

$$\begin{pmatrix} a_+ & a_- \end{pmatrix}^{-1} = \frac{1}{2\delta \grave{m}_3} \begin{pmatrix} \grave{m}_3 & 0 \\ -\grave{m}_3 & 2\delta \end{pmatrix}. \tag{5.21}$$

Furthermore by Lemma 5.10, the corresponding Floquet solutions $f_\pm = (f_1^\pm, f_2^\pm)$ of $Q(v)$ for $\lambda = \mu$ are given by

$$f_+(x,\mu) = 2\delta(\mu) M_1(x,\mu) + \grave{m}_3(\mu) M_2(x,\mu), \quad f_-(x,\mu) = \grave{m}_3(\mu) M_2(x,\mu). \tag{5.22}$$

By Lemma 5.8 and (5.19) - (5.21) one has

$$\partial_p \grave{M} = \frac{1}{8\delta \grave{m}_3} \begin{pmatrix} \grave{m}_1 \begin{pmatrix} 2\delta \\ \grave{m}_3 \end{pmatrix} & \grave{m}_4 \begin{pmatrix} 0 \\ \grave{m}_3 \end{pmatrix} \end{pmatrix} \begin{pmatrix} f_+ \cdot Z f_- & f_- \cdot Z f_- \\ -f_+ \cdot Z f_+ & -f_+ \cdot Z f_- \end{pmatrix} \frac{1}{2\delta \grave{m}_3} \begin{pmatrix} \grave{m}_3 & 0 \\ -\grave{m}_3 & 2\delta \end{pmatrix} P(\cdot) \tag{5.23}$$

Since for any 2×2 matrix $\begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \in \mathbb{C}^{2\times 2}$,

$$\begin{pmatrix} \grave{m}_1 \begin{pmatrix} 2\delta \\ \grave{m}_3 \end{pmatrix} & \grave{m}_4 \begin{pmatrix} 0 \\ \grave{m}_3 \end{pmatrix} \end{pmatrix} \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \begin{pmatrix} \grave{m}_3 & 0 \\ -\grave{m}_3 & 2\delta \end{pmatrix} = \begin{pmatrix} * & * \\ * & 2\delta \grave{m}_3 (b_2 \grave{m}_1 + b_4 \grave{m}_4) \end{pmatrix}$$

it follows from (5.23) that

$$\partial_p \grave{m}_4(\lambda)\big|_{\lambda=\mu} = \frac{1}{8\delta \grave{m}_3} (\grave{m}_1 f_- \cdot Z f_- - \grave{m}_4 f_+ \cdot Z f_-) P(\cdot).$$

By (5.22), one has

$$f_- \cdot Z f_- = \grave{m}_3^2 M_2 \cdot Z M_2, \quad f_+ \cdot Z f_- = \grave{m}_3 (2\delta M_1 + \grave{m}_3 M_2) \cdot Z M_2$$

and hence

$$\partial_p \grave{m}_4(\lambda)\big|_{\lambda=\mu} = \frac{1}{8\delta \grave{m}_3} \big((\grave{m}_1 - \grave{m}_4) \grave{m}_3^2 M_2 \cdot Z M_2 - \grave{m}_4 \grave{m}_3 2\delta M_1 \cdot Z M_2 \big) P(\cdot) = \frac{1}{4} (\grave{m}_3 M_2 \cdot Z M_2 - \grave{m}_4 M_1 \cdot Z M_2) P(\cdot).$$

Now let us turn to the computation of $\partial_q \grave{m}_4\big|_{\lambda=\mu}$. By Lemma 5.8 and (5.19)-(5.21)

$$\begin{aligned} \partial_p \grave{m}_4\big|_{\lambda=\mu} =& \frac{\mu}{4\delta \grave{m}_3} \big(\grave{m}_1 f_- \star f_- - \grave{m}_4 f_+ \star f_- \big) \\ &+ \frac{e^q}{64\mu\delta \grave{m}_3} \big(-\grave{m}_1 (f_2^-)^2 + \grave{m}_4 f_2^+ f_2^- \big) + \frac{e^{-q}}{64\mu\delta \grave{m}_3} \big(\grave{m}_1 (f_1^-)^2 - \grave{m}_4 f_1^+ f_1^- \big) \\ =& \frac{\mu}{2} \big(\grave{m}_3 M_2 \star M_2 - \grave{m}_4 M_1 \star M_2 \big) + \frac{e^q}{32\mu} \big(\grave{m}_4 m_3 m_4 - \grave{m}_3 m_4^2 \big) - \frac{e^{-q}}{32\mu} \big(\grave{m}_4 m_1 m_2 - \grave{m}_3 m_2^2 \big). \end{aligned}$$

Using that by (5.18)

$$-\grave{m}_3 \llbracket M_2 \rrbracket_{q,\mu} = \grave{m}_3 \begin{pmatrix} \frac{\mu}{2} M_2 \star M_2 - \frac{e^q}{32\mu} m_4^2 + \frac{e^{-q}}{32\mu} m_2^2 \\ \frac{1}{4} M_2 \cdot Z M_2 P(\cdot) \end{pmatrix}$$

and therefore

$$\partial_v \grave{m}_4(\lambda)\big|_{\lambda=\mu} = -\grave{m}_3 \llbracket M_2 \rrbracket_{q,\mu} + \frac{\grave{m}_4}{4} \begin{pmatrix} -2\mu M_1 \star M_2 - \frac{1}{8\mu} M_1 \cdot \begin{pmatrix} e^{-q} & \\ & -e^{q} \end{pmatrix} M_2 \\ -M_1 \cdot Z M_2 P(\cdot) \end{pmatrix}.$$

One verifies in a straightforward way that

$$\frac{\grave{m}_4}{4}\begin{pmatrix} -2\mu M_1 \star M_2 - \frac{1}{8\,\mu} M_1 \cdot \begin{pmatrix} e^{-q} & \\ & -e^{q} \end{pmatrix} M_2 \\ -M_1 \cdot Z M_2 P(\cdot) \end{pmatrix} = [\![M_1 + M_2]\!]_{q,\mu} - [\![M_1 - M_2]\!]_{q,\mu}$$

yielding the claimed identity in the case $\grave{m}_3(\mu) \neq 0$, $\delta(\mu) \neq 0$. By continuity, the identity holds for any $v \in V$. $\square$

5.3 Asymptotics

In this section we prove asymptotics for the gradients $\partial_q \grave{M}, \partial_p \grave{M}$ for bi-infinite sequences $(\zeta_n)_n \subset \mathbb{C}^*$, with $\zeta_n \sim n\pi$ as $n \to \pm\infty$. We introduce

$$\langle n \rangle := \sqrt{1 + n^2\pi^2} \tag{5.24}$$

and recall that by (2.31)

$$E_{\omega(\lambda)}(x) = \begin{pmatrix} \cos(\omega(\lambda)x) & \sin(\omega(\lambda)x) \\ -\sin(\omega(\lambda)x) & \cos(\omega(\lambda)x) \end{pmatrix}, \quad \omega(\lambda) = \lambda - \frac{1}{16\,\lambda}.$$

***Lemma* 5.15** *For any $v \in H^1_c$ and any bi-infinite sequence $(\zeta_n)_{n\in\mathbb{Z}}$ of complex numbers in $\mathbb{C}^*$ with $|\zeta_n| \geq 1/4$ the following holds:*

(i) If $\zeta_n = n\pi + O(1)$

$$\partial_q \grave{M}\Big|_{\lambda=\zeta_n} = \frac{1}{4} E_{\omega(\zeta_n)}(1) \begin{pmatrix} \cos(2\omega(\zeta_n)x) & \sin(2\omega(\zeta_n)x) \\ \sin(2\omega(\zeta_n)x) & -\cos(2\omega(\zeta_n)x) \end{pmatrix} \cdot \partial_x(\cdot) + \begin{pmatrix} \ell^2_n & \ell^2_n \\ \ell^2_n & \ell^2_n \end{pmatrix} \cdot \partial_x(\cdot) + \begin{pmatrix} \ell^2_n & \ell^2_n \\ \ell^2_n & \ell^2_n \end{pmatrix} \tag{5.25}$$

$$\partial_p \grave{M}\Big|_{\lambda=\zeta_n} = \frac{1}{4} E_{\omega(\zeta_n)}(1) \begin{pmatrix} \cos(2\omega(\zeta_n)x) & \sin(2\omega(\zeta_n)x) \\ \sin(2\omega(\zeta_n)x) & -\cos(2\omega(\zeta_n)x) \end{pmatrix} \cdot P(\cdot) + \begin{pmatrix} \ell^2_n & \ell^2_n \\ \ell^2_n & \ell^2_n \end{pmatrix} \cdot P(\cdot). \tag{5.26}$$

Alternatively, $\partial_q \grave{M}\Big|_{\lambda=\zeta_n}$ satisfies the asymptotics

$$\begin{aligned} \partial_q \grave{M} =& \frac{1}{2}\begin{pmatrix} & \sin(\omega(\zeta_n)) + \ell^2_n \\ \sin(\omega(\zeta_n)) + \ell^2_n & \end{pmatrix} E V_0 \\ & - \frac{1}{2}\zeta_n E_{\omega(\zeta_n)}(1) \begin{pmatrix} -\sin(2\omega(\zeta_n)x) & \cos(2\omega(\zeta_n)x) \\ \cos(2\omega(\zeta_n)x) & \sin(2\omega(\zeta_n)x) \end{pmatrix} + \langle n \rangle \begin{pmatrix} \ell^2_n & \ell^2_n \\ \ell^2_n & \ell^2_n \end{pmatrix} \end{aligned} \tag{5.27}$$

These estimates hold uniformly on $0 \leq x \leq 1$, on bounded subsets of H^1_c, and on subsets of bi-infinite sequences $(\zeta_n)_n$ with $\sup_{n\in\mathbb{Z}} |\zeta_n - n\pi|$ bounded. In more detail, e.g. in the first estimate the sequence of 2×2 matrices $\begin{pmatrix} a_{1n}(x) & a_{2n}(x) \\ a_{3n}(x) & a_{4n}(x) \end{pmatrix} \cdot \partial_x(\cdot)$ to be of the form $\begin{pmatrix} \ell^2_n & \ell^2_n \\ \ell^2_n & \ell^2_n \end{pmatrix} \cdot \partial_x(\cdot)$ here means that

$$\sup_{1\leq j\leq 4} \sup_{0\leq x\leq 1} \sum_{n\in\mathbb{Z}} |a_{jn}(x)|^2 \leq C$$

where the constant C can be chosen uniformly on bounded subsets H^1_c and uniformly in bi-infinite sequences $(\zeta_n)_n$ with $\sup_{n\in\mathbb{Z}} |\zeta_n - n\pi|$ bounded. In particular, $\sum_{n\in\mathbb{Z}} \|a_{j,n}\|^2 \leq C$ for any $1 \leq j \leq 4$. Furthermore, the value of $\begin{pmatrix} \ell^2_n & \ell^2_n \\ \ell^2_n & \ell^2_n \end{pmatrix} \cdot \partial_x(\cdot)$ at $\mathring{q} \in H^1_c$ is given by

$$\begin{pmatrix} \langle a_{1n}, \partial_x \mathring{q} \rangle_r & \langle a_{2n}, \partial_x \mathring{q} \rangle_r \\ \langle a_{3n}, \partial_x \mathring{q} \rangle_r & \langle a_{4n}, \partial_x \mathring{q} \rangle_r \end{pmatrix}.$$

(ii) If $\zeta_n = n\pi + \ell^2_n$, then

$$\partial_q \grave{M}\Big|_{\lambda=\zeta_n} = \frac{(-1)^n}{4} \begin{pmatrix} \cos(2\pi nx) & \sin(2\pi nx) \\ \sin(2\pi nx) & -\cos(2\pi nx) \end{pmatrix} \cdot \partial_x(\cdot) + \begin{pmatrix} \ell^2_n & \ell^2_n \\ \ell^2_n & \ell^2_n \end{pmatrix} \cdot \partial_x(\cdot) + \begin{pmatrix} \ell^2_n & \ell^2_n \\ \ell^2_n & \ell^2_n \end{pmatrix} \tag{5.28}$$

$$\partial_p \grave{M}\Big|_{\lambda=\zeta_n} = \frac{(-1)^n}{4} \begin{pmatrix} \cos(2\pi nx) & \sin(2\pi nx) \\ \sin(2\pi nx) & -\cos(2\pi nx) \end{pmatrix} \cdot P(\cdot) + \begin{pmatrix} \ell^2_n & \ell^2_n \\ \ell^2_n & \ell^2_n \end{pmatrix} \cdot P(\cdot). \tag{5.29}$$

Alternatively, $\partial_q \grave{M}\Big|_{\lambda=\zeta_n}$ *satisfies the asymptotics*

$$\partial_q \grave{M}\Big|_{\lambda=\zeta_n} = \begin{pmatrix} & \ell_n^2 \\ \ell_n^2 & \end{pmatrix} EV_0 + (-1)^{n+1}\frac{n\pi}{2}\begin{pmatrix} -\sin(2\pi nx) & \cos(\pi nx) \\ \cos(2\pi nx) & \sin(2\pi nx) \end{pmatrix} + \langle n\rangle \begin{pmatrix} \ell_n^2 & \ell_n^2 \\ \ell_n^2 & \ell_n^2 \end{pmatrix} \tag{5.30}$$

These estimates hold uniformly in $0 \le x \le 1$, on bounded subsets of H_c^1, and on subsets of bi-infinite sequences $(\zeta_n)_n$ with $(\zeta_n - n\pi)_n$ bounded in ℓ^2.

Proof. (i) Let $\zeta_n = n\pi + O(1)$. By Theorem 2.12(iii) and Corollary 2.13(iii) with $\omega(-\zeta_n) = -\omega(\zeta_n)$ one has

$$M(x)|_{\lambda=\zeta_n} = E_{\omega(\zeta_n)}(x) + \begin{pmatrix} \ell_n^2 & \ell_n^2 \\ \ell_n^2 & \ell_n^2 \end{pmatrix}, \quad M^{-1}(x)|_{\lambda=\zeta_n} = E_{-\omega(\zeta_n)}(x) + \begin{pmatrix} \ell_n^2 & \ell_n^2 \\ \ell_n^2 & \ell_n^2 \end{pmatrix}$$

uniformly in $0 \le x \le 1$. Hence (5.1) yields

$$\begin{aligned}\partial_q \grave{M} &= -\frac{1}{4}\grave{M}M^{-1}\mathrm{i}RM \cdot \partial_x(\cdot) - \frac{1}{16\,\lambda}\grave{M}M^{-1}\begin{pmatrix} & e^q \\ e^{-q} & \end{pmatrix} M \\ &= -\frac{1}{4}E_{\omega(\zeta_n)}(1)E_{-\omega(\zeta_n)}(x)\mathrm{i}RE_{\omega(\zeta_n)}(x)\cdot\partial_x(\cdot) + \begin{pmatrix} \ell_n^2 & \ell_n^2 \\ \ell_n^2 & \ell_n^2 \end{pmatrix}\cdot\partial_x(\cdot) + \begin{pmatrix} \ell_n^2 & \ell_n^2 \\ \ell_n^2 & \ell_n^2 \end{pmatrix}.\end{aligned}$$

Since by trigonometric identities

$$\begin{aligned}E_{-\omega(\zeta_n)}(x)\mathrm{i}RE_{\omega(\zeta_n)}(x) &= \begin{pmatrix} -\cos^2(\omega(\zeta_n)x) + \sin^2(\omega(\zeta_n)x) & -2\cos(\omega(\zeta_n)x)\sin(\omega(\zeta_n)x) \\ -2\cos(\omega(\zeta_n)x)\sin(\omega(\zeta_n)x) & \cos^2(\omega(\zeta_n)x) - \sin 2(\omega(\zeta_n)x) \end{pmatrix} \\ &= \begin{pmatrix} -\cos(2\omega(\zeta_n)x) & -\sin(2\omega(\zeta_n)x) \\ -\sin(2\omega(\zeta_n)x) & \cos(2\omega(\zeta_n)x) \end{pmatrix}\end{aligned}$$

estimate (5.25) follows. Alternatively by (5.3)

$$\begin{aligned}\partial_q \grave{M}\Big|_{\lambda=\zeta_n} &= \frac{1}{2}\begin{pmatrix} & \grave{m}_2 \\ -\grave{m}_3 & \end{pmatrix} EV_0 - \frac{1}{2}\grave{M}M^{-1}\Big(\zeta_n Z + \frac{1}{16\,\zeta_n}\begin{pmatrix} & e^q \\ e^{-q} & \end{pmatrix}\Big)M \\ &= \frac{1}{2}\begin{pmatrix} & \sin(\omega(\zeta_n)) + \ell_n^2 \\ \sin(\omega(\zeta_n)) + \ell_n^2 & \end{pmatrix} EV_0 - \frac{1}{2}\zeta_n E_{\omega(\zeta_n)}(1)E_{-\omega(\zeta_n)}(x)ZE_{\omega(\zeta_n)}(x) + \langle n\rangle\begin{pmatrix} \ell_n^2 & \ell_n^2 \\ \ell_n^2 & \ell_n^2 \end{pmatrix}.\end{aligned}$$

Clearly

$$\begin{aligned}E_{-\omega(\zeta_n)}(x)ZE_{\omega(\zeta_n)}(x) &= \begin{pmatrix} -2\cos(\omega(\zeta_n)x)\sin(\omega(\zeta_n)x) & \cos^2(\omega(\zeta_n)x) - \sin^2(\omega(\zeta_n)x) \\ \cos^2(\omega(\zeta_n)x) - \sin^2(\omega(\zeta_n)x) & 2\cos(\omega(\zeta_n)x)\sin(\omega(\zeta_n)x) \end{pmatrix} \\ &= \begin{pmatrix} -\sin(2\omega(\zeta_n)x) & \cos(2\omega(\zeta_n)x) \\ \cos(2\omega(\zeta_n)x) & \sin(2\omega(\zeta_n)x) \end{pmatrix}\end{aligned}$$

yielding the estimate (5.27). For $\partial_p \grave{M}$ we proceed similarly. By Proposition 5.1

$$\begin{aligned}\partial_p \grave{M} &= -\frac{\mathrm{i}}{4}\grave{M}M^{-1}RM\cdot P(\cdot) \\ &= -\frac{\mathrm{i}}{4}E_{\omega(\zeta_n)}(1)E_{-\omega(\zeta_n)}(x)RE_{\omega(\zeta_n)}(x)\cdot P(\cdot) + \begin{pmatrix} \ell_n^2 & \ell_n^2 \\ \ell_n^2 & \ell_n^2 \end{pmatrix}\cdot P(\cdot),\end{aligned}$$

leading to the estimate (5.26).
(ii) Using the estimates of Theorem 2.12(iv) and Corollary 2.13(iv) instead of the ones of Theorem 2.12(iii) and Corollary 2.13(iii) in the proof above, one obtains estimates (5.28), (5.29), and (5.30). Going through the arguments of the proof one sees that the claimed uniformity statements hold. □

The asymptotics stated in Lemma 5.15 lead to asymptotic estimates of the gradients of Δ and δ, introduced in (2.33).

***Corollary* 5.16** *For any $v \in H_c^1$ and any bi-infinite sequence of complex numbers $\zeta_n = n\pi + O(1)$*

$$(\partial_q\Delta, \partial_p\Delta)\Big|_{\lambda=\zeta_n} = \big(\ell_n^2\cdot\partial_x(\cdot) + \ell_n^2,\ \ell_n^2\cdot P(\cdot)\big)$$

and, alternatively, $\partial_q \Delta\Big|_{\lambda=\zeta_n} = \langle n\rangle \ell_n^2$. *Furthermore*

$$(\partial_q\delta, \partial_p\delta)\Big|_{\lambda=\zeta_n} = \left(\frac{1}{4}\cos(\omega(\zeta_n)(1-2x))\cdot\partial_x(\cdot) + \ell_n^2\cdot\partial_x(\cdot) + \ell_n^2, \quad \frac{1}{4}\cos(\omega(\zeta_n)(1-2x))\cdot P(\cdot) + \ell_n^2\cdot P(\cdot)\right).$$

Alternatively, the asymptotics of $\partial_q\delta\Big|_{\lambda=\zeta_n}$ *can be written as*

$$\partial_q\delta\Big|_{\lambda=\zeta_n} = -\frac{\zeta_n}{2}\sin(\omega(\zeta_n)(1-2x)) + \langle n\rangle\ell_n^2.$$

These estimates hold uniformly in $0 \le x \le 1$, *on bounded subsets of* H_c^1, *and on subsets of bi-infinite sequences* $(\zeta_n)_n$ *with* $\sup_{n\in\mathbb{Z}}|\zeta_n - n\pi|$ *bounded. If furthermore* $\zeta_n = n\pi + \ell_n^2$ *then*

$$(\partial_q\delta, \partial_p\delta)\Big|_{\lambda=\zeta_n} = (-1)^n\left(\frac{1}{4}\cos(2n\pi x)\cdot\partial_x(\cdot) + \ell_n^2\cdot\partial_x(\cdot) + \ell_n^2, \quad \frac{1}{4}\cos(2n\pi x)\cdot P(\cdot) + \ell_n^2\cdot P(\cdot)\right).$$

Alternatively, the asymptotics of $\partial_q\delta\Big|_{\lambda=\zeta_n}$ *can be written as*

$$\partial_q\delta\Big|_{\lambda=\zeta_n} = (-1)^n\frac{n\pi}{2}\sin(2\pi nx) + \langle n\rangle\ell_n^2$$

uniformly in $0 \le x \le 1$, *on bounded subsets of* H_c^1, *and on subsets of bi-infinite sequences* $(\zeta_n)_n$ *with* $(\zeta_n - n\pi)_n$ *bounded in* ℓ^2.

Proof. By (5.25)

$$\partial_q\dot{M}\Big|_{\lambda=\zeta_n} = \frac{1}{4}E_{\omega(\zeta_n)}(1)\begin{pmatrix}\cos(2\omega(\zeta_n)x) & \sin(2\omega(\zeta_n)x)\\ \sin(2\omega(\zeta_n)x) & -\cos(2\omega(\zeta_n)x)\end{pmatrix}\cdot\partial_x(\cdot) + \begin{pmatrix}\ell_n^2 & \ell_n^2\\ \ell_n^2 & \ell_n^2\end{pmatrix}\cdot\partial_x(\cdot) + \begin{pmatrix}\ell_n^2 & \ell_n^2\\ \ell_n^2 & \ell_n^2\end{pmatrix}$$

and by (5.27) one has

$$\begin{aligned}\partial_q\dot{M}\Big|_{\lambda=\zeta_n} =& \frac{1}{2}\begin{pmatrix} & \sin(\omega(\zeta_n)) + \ell_n^2\\ \sin(\omega(\zeta_n)) + \ell_n^2 & \end{pmatrix}EV_0\\ &- \frac{1}{2}\zeta_n E_{\omega(\zeta_n)}(1)\begin{pmatrix}-\sin(2\omega(\zeta_n)x) & \cos(2\omega(\zeta_n)x)\\ \cos(2\omega(\zeta_n)x) & \sin(2\omega(\zeta_n)x)\end{pmatrix} + \langle n\rangle\begin{pmatrix}\ell_n^2 & \ell_n^2\\ \ell_n^2 & \ell_n^2\end{pmatrix}.\end{aligned}$$

Furthermore by (5.26)

$$\partial_p\dot{M}\Big|_{\lambda=\zeta_n} = \frac{1}{4}E_{\omega(\zeta_n)}(1)\begin{pmatrix}\cos(2\omega(\zeta_n)x) & \sin(2\omega(\zeta_n)x)\\ \sin(2\omega(\zeta_n)x) & -\cos(2\omega(\zeta_n)x)\end{pmatrix}\cdot P(\cdot) + \begin{pmatrix}\ell_n^2 & \ell_n^2\\ \ell_n^2 & \ell_n^2\end{pmatrix}\cdot P(\cdot).$$

Since

$$E_{\omega(\zeta_n)}(1)\begin{pmatrix}-\sin(2\omega(\zeta_n)x) & \cos(2\omega(\zeta_n)x)\\ \cos(2\omega(\zeta_n)x) & \sin(2\omega(\zeta_n)x)\end{pmatrix} = \begin{pmatrix}\sin(\omega(\zeta_n)(1-2x)) & \cos(\omega(\zeta_n)(1-2x))\\ \cos(\omega(\zeta_n)(1-2x)) & -\sin(\omega(\zeta_n)(1-2x))\end{pmatrix}$$

and

$$E_{\omega(\zeta_n)}(1)\begin{pmatrix}\cos(2\omega(\zeta_n)x) & \sin(2\omega(\zeta_n)x)\\ \sin(2\omega(\zeta_n)x) & -\cos(2\omega(\zeta_n)x)\end{pmatrix} = \begin{pmatrix}\cos(\omega(\zeta_n)(1-2x)) & -\sin(\omega(\zeta_n)(1-2x))\\ -\sin(\omega(\zeta_n)(1-2x)) & -\cos(\omega(\zeta_n)(1-2x))\end{pmatrix}$$

the claimed estimates hold. The uniformity statements follow from Lemma 5.15.

□

Lemma 5.17 *For any* $v_0 \in H_c^1$ *there exists a neighborhood* V *of* v_0 *in* H_c^1 *and* $N \ge 1$ *such that for any* $|n| \ge N$ *and* $v \in V$, $Q(v)$ *has precisely one Dirichlet eigenvalue in* D_n, *denoted by* μ_n. *Then* μ_n *is analytic on* V *for* $|n| \ge N$ *and as* $n \to \infty$

$$(\partial_q\mu_n, \partial_p\mu_n) = -\frac{1}{4}\Big(\sin(2n\pi x)\cdot\partial_x(\cdot) + \ell_n^2\cdot\partial_x(\cdot) + \ell_n^2,\ \sin(2n\pi x)\cdot P(\cdot) + \ell_n^2\cdot P(\cdot)\Big). \qquad (5.31)$$

Integrating by parts, the asymptotics of $\partial_q\mu_n$ take the form

$$\partial_q\mu_n = \frac{n\pi}{2}\cos(2n\pi x) + \langle n\rangle\ell_n^2. \tag{5.32}$$

These estimates hold uniformly on $0 \le x \le 1$ and on bounded subsets of V. In more detail e.g. the term $\ell_n^2 \cdot \partial_x(\cdot)$ in the expression for $\partial_q\mu_n$ means that there exist functions $a_n(x)$, $|n| > N$, so that for some $C > 0$

$$\sup_{0\le x\le 1}\sum_{|n|\ge N}|a_n(x)| \le C$$

where C can be chosen uniformly on bounded subsets of V. Furthermore the value of $\ell_n^2 \cdot \partial_x$ at $\mathring{q}$ is $\langle a_n, \partial_x q\rangle_r$.

Proof. Take V and N to be as in Lemma 3.4. Then μ_n is simple for $|n| > N$ and hence analytic by Lemma 5.6. Furthermore, by Lemma 3.16, $\mu_n = n\pi + \ell_n^2$, implying that by Theorem 2.12(iv) for $n \to \infty$

$$M(x,\mu_n,q,p) = E_{n\pi}(x) + \ell_n^2, \qquad \dot{M}(x,\mu_n,q,p) = xJE_{n\pi}(x) + \ell_n^2,$$

where we recall that by (2.31)

$$E_{n\pi}(x) = \begin{pmatrix}\cos(n\pi x) & \sin(n\pi x)\\ -\sin(n\pi x) & \cos(n\pi x)\end{pmatrix}.$$

In particular, $\dot{\chi}_D(\mu_n) = \dot{m}_2(\mu_n) = (-1)^n + \ell_n^2$, $\grave{m}_1(\mu_n) = (-1)^n + \ell_n^2$. Hence by Lemma 5.6, for $\mathring{v} = (\mathring{q},\mathring{p}) \in H_c^1$

$$\mathrm{d}\mu[\mathring{v}]_n = \frac{\grave{m}_1}{\dot{\chi}_D}\int_0^1 [\![M_2]\!]_{q,\mu}\begin{pmatrix}\mathring{q}\\ \mathring{p}\end{pmatrix}\mathrm{d}x = \int_0^1 \left[\!\!\left[\begin{pmatrix}\sin(n\pi x)+\ell_n^2\\ \cos(n\pi x)+\ell_n^2\end{pmatrix}\right]\!\!\right]_{q,\mu}\begin{pmatrix}\mathring{q}\\ \mathring{p}\end{pmatrix}\mathrm{d}x.$$

Since by definition (5.10)

$$\left[\!\!\left[\begin{pmatrix}\sin(n\pi x)+\ell_n^2\\ \cos(n\pi x)+\ell_n^2\end{pmatrix}\right]\!\!\right]_{q,\mu} = \begin{pmatrix}\frac{\mu_n}{2}(\cos^2(n\pi x) - \sin^2(n\pi x) + \ell_n^2) + \frac{1}{32\mu_n}(\cos^2(n\pi x)e^q - \sin^2(n\pi x)e^{-q}) + \ell_n^2\\ -\frac{1}{2}\sin(n\pi x)\cos(n\pi x)\cdot P(\cdot) + \ell_n^2\cdot P(\cdot)\end{pmatrix}$$

it then follows that

$$\mathrm{d}\mu_n[\mathring{v}] = \begin{pmatrix}\frac{n\pi}{2}\cos(2n\pi x) + \langle n\rangle\ell_n^2 + \ell_n^2\\ -\frac{1}{4}\sin(2n\pi x)\cdot P(\cdot) + \ell_n^2\cdot P(\cdot)\end{pmatrix}$$

yielding (5.32). On the other hand by Lemma 5.4

$$\begin{aligned}\mathrm{d}\mu_n[\mathring{v}] &= \frac{\grave{m}_1}{\dot{\chi}_D}\int_0^1 \frac{1}{16\mu_n}(m_4^2e^q - m_2^2e^{-q})\mathring{q} - \frac{1}{2}(P(\mathring{p}) + \partial_x\mathring{q})m_2m_4\,\mathrm{d}x\\ &= \int_0^1 \begin{pmatrix}\ell_n^2 - \frac{1}{2}\sin(n\pi x)\cos(n\pi x)\cdot\partial_x(\cdot) + \ell_n^2\cdot\partial_x(\cdot) + \ell_n^2\\ -\frac{1}{2}\sin(n\pi x)\cos(n\pi x)\cdot P(\cdot) + \ell_n^2\cdot P(\cdot)\end{pmatrix}\cdot\begin{pmatrix}\mathring{q}\\ \mathring{p}\end{pmatrix}\mathrm{d}x\\ &= -\frac{1}{4}\int_0^1 \begin{pmatrix}\sin(2n\pi x)\cdot\partial_x(\cdot) + \ell_n^2\cdot\partial_x(\cdot) + \ell_n^2\\ \sin(2n\pi x)\cdot P(\cdot) + \ell_n^2\cdot P(\cdot)\end{pmatrix}\cdot\begin{pmatrix}\mathring{q}\\ \mathring{p}\end{pmatrix}\mathrm{d}x\end{aligned}$$

which proves (5.31). □

6 Real and almost real potentials

In this chapter we discuss the periodic and the Dirichlet spectrum of $Q(v)$ in the case where the potentials v are real or close to real ones and introduce the notion of isolating neighbourhoods. The space of real potentials v is denoted by H_r^1. More generally, for any $m \ge 1$ let

$$H_r^m := H^m(\mathbb{T},\mathbb{R}) \times H^m(\mathbb{T},\mathbb{R}) \subset H_c^1, \qquad \forall\, m \ge 1.$$

Without further reference we use the notation introduced in Chapter 3. In particular, we recall that the sets D_n, B_n, A_n are defined in (3.1)-(3.3). Furthermore we recall that for any $v \in H_c^1$ and $n \in \mathbb{Z}$, $G_n = [\lambda_n^-, \lambda_n^+]$ denotes the n'th closed gap and $\gamma_n = \lambda_n^+ - \lambda_n^-$ is referred to as the n'th gap length.

6.1 Real potentials

We have seen that for $v \in H^1_c$, one has $\Delta(\lambda^{\pm}_n) = (-1)^n$ for $|n|$ big enough, but that such an identity not necessarily holds for $|n|$ small. In the case $v \in H^1_r$, the operator $Q(v)$ with periodic boundary conditions is self-adjoint, hence the periodic spectrum is real and the listing (3.18) of the periodic eigenvalues in $\mathbb{C}^+$ coincides with the one obtained from the ordering on $\mathbb{R}$,

$$0 < \cdots \le \lambda^-_{-1} \le \lambda^+_{-1} \le \lambda^-_0 \le \lambda^+_0 \le \lambda^-_1 \le \lambda^+_1 \le \cdots .$$

Actually, in this case more can be said. First we note that the periodic eigenvalues λ^+_n, λ^-_n $n \in \mathbb{Z}$, are continuous in v on H^1_r (cf. Proposition 6.3 below). Since for $v = 0$, $\Delta(\lambda^{\pm}_n) = (-1)^n$ it then follows by deforming an arbitrary element $v \in H^1_r$ along the straight line $t \mapsto t \cdot v$ that on H^1_r, $\Delta(\lambda^{\pm}_n) = (-1)^n$ for any $n \in \mathbb{Z}$. Hence the periodic eigenvalues $\lambda^{\pm}_n$, $n \in \mathbb{Z}$, of $Q(v)$ in $\mathbb{C}^+$ satisfy

$$0 < \cdots < \lambda^-_{-1} \le \lambda^+_{-1} < \lambda^-_0 \le \lambda^+_0 < \lambda^-_1 \le \lambda^+_1 < \cdots$$

with strict inequality between λ^+_n and λ^-_{n+1} for any $n \in \mathbb{Z}$. Similarly, for $v \in H^1_r$ the operator $Q(v)$ with Dirichlet boundary conditions is self-adjoint and hence the Dirichlet spectrum is real. It turns out that the Dirichlet eigenvalues in $\mathbb{C}^+$ can be located as follows.

***Lemma* 6.1** *For any $v \in H^1_r$ and $n \in \mathbb{Z}$, the Dirichlet eigenvalue μ_n is real and satisfies $\lambda^-_n \le \mu_n \le \lambda^+_n$. Furthermore, the Dirichlet eigenvalues are real analytic functions on H^1_r.*

Proof. Let $v \in H^1_r$. By Lemma 3.8, for any Dirichlet eigenvalue μ_n, one has $\Delta^2(\mu_n) - 1 = \delta^2(\mu_n) \ge 0$, hence $|\Delta(\mu_n)| \ge 1$. Since $\mu_n(0) = \lambda^+_n(0)$ one then sees that $\lambda^-_n \le \mu_n \le \lambda^+_n$ by deforming v to 0 along the straight line $t \mapsto tv$. In particular it follows that any Dirichlet eigenvalue is simple. Hence by Lemma 5.6, it is real analytic. □

Next let us consider the roots of $\dot\Delta$ in the case where $v \in H^1_r$. For $v = 0$ the roots in $\mathbb{C}^+$ consist of a bi-infinite sequence

$$0 < \cdots < \dot\lambda_{-2} < \dot\lambda_{-1} < \dot\lambda_0 < \dot\lambda_1 < \cdots$$

and the additional root $\dot\lambda_* = \frac{\mathrm{i}}{4}$. For arbitrary $v \in H^1_r$ the following holds.

***Lemma* 6.2** *For any $v \in H^1_r$, the roots of $\dot\Delta$ in $\mathbb{C}^+$ are all simple and can be listed in form of a bi-infinite sequence of real numbers*

$$0 < \cdots < \dot\lambda_{-2} < \dot\lambda_{-1} < \dot\lambda_0 < \dot\lambda_1 < \dot\lambda_2 < \cdots$$

and one additional purely imaginary root $\dot\lambda_* \in \mathrm{i}\mathbb{R}_{>0}$. Furthermore if $\lambda^-_n = \lambda^+_n$ for any given $n \in \mathbb{Z}$, then $\lambda^-_n = \dot\lambda_n$ whereas if $\lambda^-_n < \lambda^+_n$ then $\lambda^-_n < \dot\lambda_n < \lambda^+_n$. Furthermore, $\dot\lambda_n$, $n \in \mathbb{Z}$, and $\dot\lambda_*$ are real analytic on H^1_r.

Proof. Let $v \in H^1_r$. By Lemma 2.14(i) $\Delta(-\lambda, v) = \Delta(\lambda, v)$ and by Lemma 2.14(iii) $\overline{\Delta(\overline{\lambda}, v)} = \Delta(\lambda, v)$. This implies that $\dot\Delta(-\lambda, v) = -\dot\Delta(\lambda, v)$ and $\overline{\dot\Delta(\overline{\lambda}, v)} = \dot\Delta(\lambda, v)$. Hence in particular $\Delta(\lambda, v)$ is real for $\lambda \in \mathbb{R}$ and for any root $\dot\lambda \in \mathbb{C}^*$ of $\dot\Delta$, the complex numbers $-\dot\lambda$, $\overline{\dot\lambda}$, and $-\overline{\dot\lambda}$ are also roots of $\dot\Delta$. Furthermore, since $\Delta(\lambda, 0) \in (-1, 1)$ for any $\lambda \in (\lambda^+_n(0), \lambda^-_{n+1}(0))$, it follows by continuity that $\Delta(\lambda, v) \in (-1, 1)$ for $\lambda \in (\lambda^+_n(v), \lambda^-_{n+1}(v))$. If the n'th gap is collapsed then $\Delta^2 - 1$ has a root of multiplicity two at $\lambda^+_n = \lambda^-_n$. Since $\Delta(\lambda^{\pm}_n) \ne 0$ it then follows that λ^+_n is also a root of $\dot\Delta$. If $\lambda^-_n \ne \lambda^+_n$, both roots have algebraic multiplicity one. Since $\Delta(\lambda)$ is real valued on $\mathbb{R}$, $-1 < \Delta(\lambda) < 1$ for $\lambda^+_{n-1} < \lambda < \lambda^-_n$, $\Delta(\lambda^-_n) = \Delta(\lambda^+_n) = (-1)^n$ and $(-1)^n\dot\Delta(\lambda^-_n) > 0$ it follows that $|\Delta(\lambda)| > 1$ for all $\lambda \in (\lambda^-_n, \lambda^+_n)$. This implies that $|\Delta(\lambda)|$ has a maximum in $(\lambda^-_n, \lambda^+_n)$. Hence $\dot\Delta(\lambda)$ vanishes at least once in G_n for any $n \in \mathbb{Z}$. Choose such a root and denote it by $\dot\lambda_n$. By the previous discussion $\dot\lambda_n$ is real and $-\dot\lambda_n$ is another root of $\dot\Delta$. By the Counting Lemma (Lemma 3.12) there exists $N \ge 1$ such that for any $n > N$, $\dot\Delta$ has exactly one simple root in each of the domains D_n, $-D_n$, D_{-n}, and $-D_{-n}$ and $4 + 4N$ roots in the annulus A_N. Since $G_n \subset D_n$ and $G_{-n} \subset D_{-n}$ for any $n > N$, the roots in D_n, $-D_n$, D_{-n}, and $-D_{-n}$ are $\dot\lambda_n, -\dot\lambda_n, \dot\lambda_{-n}$, and $-\dot\lambda_{-n}$, respectively. Furthermore $\dot\lambda_0, -\dot\lambda_0 \in A_N$ and for any $1 \le n \le N$, one has $\dot\lambda_n, -\dot\lambda_n, \dot\lambda_{-n}, -\dot\lambda_{-n} \in A_N$. Hence in addition to $\dot\lambda_n, -\dot\lambda_n$, $n \in \mathbb{Z}$, $\dot\Delta$ has two more roots $\nu_1, \nu_2 \in A_N$. Since $\dot\lambda_n, -\dot\lambda_n$, $n \in \mathbb{Z}$, are all simple roots of $\dot\Delta$, $\nu_1, \nu_2 \notin \{\, \dot\lambda_n, -\dot\lambda_n \,:\, n \in \mathbb{Z} \,\}$. By the previous discussion $-\nu_1, -\nu_2$ as well as $\overline{\nu_1}, \overline{\nu_2}$ are also roots of $\dot\Delta$. The Counting Lemma then implies that $\nu_1 = -\nu_2$ and $\nu_1 \in (\mathbb{R} \cup \mathrm{i}\mathbb{R}) \cap A_N \subset (\mathbb{R} \cup \mathrm{i}\mathbb{R}) \setminus \{0\}$. In particular it follows that $\nu_1 \ne \nu_2$ and one of the two roots, denoted by $\dot\lambda_*$, is in $\mathbb{C}^+$. At $v = 0$, $\dot\lambda_*(0) = \frac{\mathrm{i}}{4}$ by Lemma 2.16. Taking into account that the G_n's are disjoint, one sees that all roots of $\dot\Delta$ are simple on H^1_r. Since the roots of $\dot\Delta$ are all simple and $\dot\Delta(\lambda, v)$ is real analytic in $\mathbb{C}^* \times H^1_r$, it follows by the argument principle that these roots are real analytic. □

We finish this section with an additional result needed for the construction of Birkhoff coordinates.

***Proposition* 6.3** *The Dirichlet, Neumann, and periodic eigenvalues as well as the roots of $\dot{\Delta}$ are compact functions on H_r^1.*

Proof. By Proposition 2.5 and Proposition 2.6, on any closed bounded subset of $[0,\infty) \times \mathbb{C}^* \times H_r^1$, $M(x,\lambda,v)$ and $\dot{M}(x,\lambda,v)$ are compact functions and hence so are χ_p, χ_D, and $\dot{\Delta}$ on closed, bounded subsets of $\mathbb{C}^* \times H_r^1$. Recall that for any $v \in H_r^1$, the roots of χ_D and $\dot{\Delta}$ are simple and the ones of χ_p at most double. Since the case of a simple root is simpler, let us only treat the double roots of χ_p. Assume that $(v_k)_{k\geq 1} \subset H_r^1$ with $v_k \rightharpoonup v_*$ and that $\kappa_* := \lambda_n^+(v_*) = \lambda_n^-(v_*)$ for some $n \in \mathbb{Z}$. To simplify notation introduce $f_k(\lambda) := \chi_p(\lambda, v_k)$, $f_*(\lambda) := \chi_p(\lambda, v_*)$. Choose a disc D, centered at κ_*, of sufficiently small radius $\rho > 0$ so that $\min_{\lambda\in\partial D} |f_*(\lambda)| \geq 2\epsilon$. Since ∂D is compact, f_k, f_* are analytic and $\lim_{k\to\infty} f_k(\lambda) = f_*(\lambda)$ for any $\lambda \in \overline{D}$, there exists $k_0 \geq 1$ so that $\min_{\lambda\in\partial D} |f_k(\lambda)| \geq \epsilon$ for any $k \geq k_0$. It then follows from the argument principle that for any $k \geq k_0$, $f_*(\lambda)$ has precisely two roots (counted with multiplicities) inside D, i.e., for k sufficiently large

$$\int_{\partial D} \frac{\dot{f}_k(\lambda)}{f_k(\lambda)}\, \mathrm{d}\lambda = \int_{\partial D} \frac{\dot{f}_*(\lambda)}{f_*(\lambda)}\, \mathrm{d}\lambda = 2.$$

Using the Counting Lemma and again the argument principle one sees that for k sufficiently large these roots of $f_k(\lambda)$ coincide with the pair $\lambda_n^+(v_k)$, $\lambda_n^-(v_k)$. Since the radius $\rho > 0$ of the disc D can be chosen arbitrarily small it then follows that $\lim_{k\to\infty} \lambda_n^\pm(v_k) = \lambda_n^\pm(v_*)$. □

For the construction of Birkhoff coordinates the sequences $(\gamma_n)_{n\in\mathbb{Z}}, (\tau_n)_{n\in\mathbb{Z}}$, defined on H_r^1 by

$$\gamma_n = \lambda_n^+ - \lambda_n^-, \quad \tau_n = (\lambda_n^+ + \lambda_n^-)/2 \tag{6.1}$$

will play an important role. The quantity γ_n is referred to as the n'th gap length. By Proposition 6.3 γ_n and τ_n are continuous functions on H_r^1.

6.2 Almost real potentials

In order to construct real analytic Birkhoff coordinates on H_r^1 we need to consider potentials in some complex neighborhood of H_r^1 in H_c^1. It is important to choose this neighborhood sufficiently small so that the Dirichlet eigenvalues and the roots of $\dot{\Delta}$ remain simple and can be localized sufficiently accurately in terms of isolating neighborhoods, which we now introduce.

First we need to discuss how to list all of the roots of $\dot{\Delta}$ in a convenient way. Since by Lemma 6.2 all roots of $\dot{\Delta}$ are real analytic and simple on H_r^1 it follows from the argument principle that any root of $\dot{\Delta}$ extends analytically to some (small) neighborhood of H_r^1 in H_c^1 and remains simple. Furthermore recall that by (3.20) (Counting Lemma), for any $v \in H_c^1$ there exists $N \geq 1$ so that the roots of $\dot{\Delta}(\lambda)$ in $\mathbb{C}^+ \setminus A_N$ can be listed as a bi-infinite sequence

$$0 \preceq \cdots \preceq \dot{\lambda}_{-N-2} \preceq \dot{\lambda}_{-N-1} \preceq \dot{\lambda}_{N+1} \preceq \dot{\lambda}_{N+2} \preceq \cdots$$

with $\dot{\lambda}_n \in D_n \ \forall\, |n| > N$ and any root $\dot{\lambda}$ of the $2N+2$ remaining ones in $\mathbb{C}^+$ satisfies $\dot{\lambda}_{-N-1} \preceq \dot{\lambda} \preceq \dot{\lambda}_{N+1}$. Hence for any element $v_0 \in H_r^1$ there exists a neighborhood V in H_c^1 so that for any $v \in V$ the roots of $\dot{\Delta}(\lambda)$ are simple and and the root $\dot{\lambda}_*(v)$ denotes the one obtained from $\dot{\lambda}_*(v_0)$ by analytic deformation. Note that $\dot{\lambda}_*(v)$ is not necessarily in $\mathbb{C}^+$. We list the roots which are different from $\dot{\lambda}_*(v)$ and $-\dot{\lambda}_*(v)$ and are contained in $\mathbb{C}^+$ as a bi-infinite sequence,

$$0 \preceq \cdots \preceq \dot{\lambda}_{-2} \preceq \dot{\lambda}_{-1} \preceq \dot{\lambda}_0 \preceq \dot{\lambda}_1 \preceq \dot{\lambda}_2 \preceq \cdots, \tag{6.2}$$

so that

$$\dot{\lambda}_n = n\pi + o(1), \qquad \frac{1}{16\dot{\lambda}_{-n}} = n\pi + o(1) \quad n \to \infty.$$

Making the neighborhood V smaller if needed we can assume that the additional root $\dot{\lambda}_*$ satisfies

$$\inf\{\, \mathrm{Im}\dot{\lambda}_*(v) : v \in V \,\} > 0.$$

For a potential v in V we say that a sequence $(U_n)_{n\in\mathbb{Z}}$ of pairwise disjoint open neighborhoods in $\mathbb{C}^+$ together with an open disc $U_* \subset \{\, \lambda \in \mathbb{C} : \mathrm{Im}\lambda > 0 \,\}$, centered on $\mathrm{i}\mathbb{R}_{>0}$, form a set of isolating neighborhoods if the following properties hold:

(I-1) $G_n \subset U_n \subset \mathbb{C}^+$ and $\mu_n, \dot{\lambda}_n \in U_n$ for any $n \in \mathbb{Z}$, $\dot{\lambda}_* \in U_*$.

(I-2) For any $n \geq 0$, U_n is a disc centered on the real axis so that there exists a constant $c \geq 1$ with

$$c^{-1}|m-n| \leq dist(U_m, U_n) \leq c|m-n| \quad \forall\, m,n \geq 0,\, m \neq n.$$

(I-3) The sets $\{ \frac{1}{16\lambda} \,:\, \lambda \in U_{-n} \}$, $n \geq 0$, satisfy (I-2) with the same constant c.

(I-4) For $|n|$ sufficiently large $U_n = D_n$.

(I-5) $c^{-1} \leq \text{dist}(U_*, U_n)\ \ \forall\, n \in \mathbb{Z}$ with $c \geq 1$ given as in (I-2).

Note that if for a potential $v \in V$ such a set of isolating neighborhoods exists, then the Dirichlet eigenvalues of $Q(v)$ and the roots of $\dot{\Delta}(\cdot, v)$ are all simple.

***Lemma* 6.4** *Let $(U_n)_{n\in\mathbb{Z}}$, U_* be a set of isolating neighborhoods of v_0 in V. Then there exists a neighborhood $V_{v_0} \subset V$ of v_0 in H^1_c such that it is a set of isolating neighborhoods for any v in V_{v_0}.*

Proof. By Lemma 3.4, Lemma 3.11, Lemma 3.12 (counting lemmas), there exist an integer $N \geq 1$ and a neighborhood $V_{v_0} \subset V$ of v_0 in H^1_c such that for any v in V_{v_0},

$$G_n(v) \subset D_n, \quad \mu_n, \dot{\lambda}_n \in D_n, \quad U_n = D_n, \quad |n| \geq N.$$

Clearly (I-2)-(I-5) are satisfied and so it suffices to verify that (I-1) holds on V_{v_0}, possibly after shrinking it if needed. It remains to control finitely many spectral quantities. Since χ_D is analytic in λ and v and by (I-1) does not vanish on ∂U_n when evaluated at v_0, one has, possibly after shrinking V_{v_0} if needed, that for any $v \in V_{v_0}$ and $|n| < N$

$$|\chi_D(\lambda, v) - \chi_D(\lambda, v_0)| < |\chi_D(\lambda, v_0)| \quad \forall\, \lambda \in \partial U_n. \tag{6.3}$$

Hence by Rouché's theorem $\chi_D(\cdot, v)$ has the same number of roots inside U_n as $\chi_D(\cdot, v_0)$ for any $|n| < N$. Similarly one argues for χ_p and $\dot{\Delta}$, implying altogether that (I-1) holds for v in V_{v_0}, possibly after shrinking it once more if needed. □

For real potentials, one has for any $n \in \mathbb{Z}$, $\lambda^+_n < \lambda^-_{n+1}$, $\lambda^-_n \leq \dot{\lambda}_n \leq \lambda^+_n$, $\dot{\lambda}_* \in \mathrm{i}\mathbb{R}_{>0}$ (Lemma 6.2) and $\lambda^-_n \leq \mu_n \leq \lambda^+_n$ (Lemma 6.1) and hence a set of isolating neighborhoods always exists. By Lemma 6.4 it follows that for any $v \in H^1_r$, there is a (small) neighborhood $V_v \subset V$ in H^1_c and subsets $(U_n)_{n\in\mathbb{Z}}$, U_* so that they are isolating neighborhoods for any potential in V_v. Setting

$$\hat{W} := \bigcup_{v \in H^1_r} V_v, \tag{6.4}$$

we thus obtain an open connected neighborhood of H^1_r in H^1_c. Without loss of generality we can assume that $v \in \hat{W}$ if and only if $(-q, p) \in \hat{W}$.

***Lemma* 6.5** *On $\hat{W}$, the Dirichlet eigenvalues μ_n, $n \in \mathbb{Z}$, are real analytic functions of q and p. For $v = (q, p) \in \hat{W}$ one has*

$$\frac{1}{16\mu_n(q,p)} = \mu_{-n}(-q, p) \quad \forall\, n \in \mathbb{Z}. \tag{6.5}$$

Proof. The real analyticity follows by the fact that all Dirichlet eigenvalues are simple on $\hat{W}$. Concerning (6.5) recall that the Dirichlet eigenvalues are the zeros of $\grave{m}_2(\lambda, q, p)$ and that by Proposition 2.1, $\grave{m}_2(\frac{1}{16\lambda}, q, p) = -e^{-q(0)}\grave{m}_2(\lambda, -q, p)$. Hence for any $\mu \in \mathbb{C}^*$

$$\mu \in \text{spec}_{dir} Q(q,p) \quad \Leftrightarrow \quad \frac{1}{16\,\mu} \in \text{spec}_{dir} Q(-q, p). \tag{6.6}$$

It remains to show that the listing of the Dirichlet eigenvalues of $Q(q, p)$ and $Q(-q, p)$ lead to (6.5). Taking $N \geq 1$ as in Lemma 3.4 (Counting Lemma) and noting that by definition (3.1), for any $\lambda \in D_{-n}$ $n \geq 1$, $\frac{1}{16\lambda} \in D_n$, one has

$$\frac{1}{16\mu_n(q,p)} = \mu_{-n}(-q, p) \quad \text{for all} \quad |n| > N. \tag{6.7}$$

For the finitely many Dirichlet eigenvalues μ_n with $|n| \le N$, note that by the definition of isolating neighborhoods, $\mathrm{Re}\mu_n(q,p) > 0$ for all n and by (3.11) for any $a, b \in \{ \lambda : \mathrm{Re}\lambda > 0 \}$

$$a \preceq b \quad \Leftrightarrow \quad \frac{1}{16b} \preceq \frac{1}{16a}.$$

Hence

$$\frac{1}{16\,\mu_N(q,p)} \preceq \frac{1}{16\,\mu_{N-1}(q,p)} \preceq \cdots \preceq \frac{1}{16\,\mu_{-(N-1)}(q,p)} \preceq \frac{1}{16\,\mu_{-N}(q,p)}$$

Since

$$\mu_{-N}(-q,p) \preceq \mu_{-(N-1)}(-q,p) \preceq \cdots \preceq \mu_N(-q,p)$$

it then follows from (6.7), that also for any $|n| \le N$, one has $\frac{1}{16\mu_n(q,p)} = \mu_{-n}(-q,p)$. □

Lemma 6.6 (i) *On $\hat{W}$, $\dot{\lambda}_n$, $n \in \mathbb{Z}$, and $\mathrm{i}\dot{\lambda}_*$ are real analytic functions. For $(q,p) \in \hat{W}$ one has*

$$\frac{1}{16\,\dot{\lambda}_*(q,p)} = -\dot{\lambda}_*(-q,p) \quad \textit{and} \quad \frac{1}{16\,\dot{\lambda}_n(q,p)} = \dot{\lambda}_{-n}(-q,p).$$

(ii) *For any $(q,p) \in \hat{W}$,*

$$\frac{1}{16\,\lambda_n^{\pm}(q,p)} = \lambda_{-n}^{\mp}(-q,p) \quad \forall\, n \in \mathbb{Z}.$$

Proof. One argues as in the proof of Lemma 6.5. We only remark that for $\dot{\lambda}_*(q,p)$ the identity $\frac{1}{16\dot{\lambda}_*(q,p)} = -\dot{\lambda}_*(-q,p)$ holds since among the two roots $\dot{\lambda}_*(-q,p)$ and $-\dot{\lambda}_*(-q,p)$, the root $\dot{\lambda}_*(-q,p)$ is characterized by $\mathrm{Im}\dot{\lambda}_*(-q,p) > 0$. □

We now analyze the following quantities in more detail,

$$\tau_n = \frac{\lambda_n^+ + \lambda_n^-}{2}, \quad \gamma_n = \lambda_n^+ - \lambda_n^-, \quad n \in \mathbb{Z}. \tag{6.8}$$

First we need to establish the following auxiliary result.

Lemma 6.7 *For any $k \ge 1$ and $n \in \mathbb{Z}$ the functions $(\lambda_n^+)^k + (\lambda_n^-)^k$ are analytic on $\hat{W}$ and real on H_r^1.*

Proof. Let v be in $\hat{W}$ and let $(U_n)_{n\in\mathbb{Z}}$, U_* be a set of isolating neighborhoods for the neighborhood V_v provided by Lemma 6.4. Then for any $k \ge 1$ and $n \in \mathbb{Z}$, it follows from the argument principle that

$$(\lambda_n^+)^k + (\lambda_n^-)^k = \frac{1}{2\pi\mathrm{i}} \int_{\partial U_n} \lambda^k \frac{2\Delta(\lambda)\dot{\Delta}(\lambda)}{\Delta^2(\lambda) - 1}\, \mathrm{d}\lambda. \tag{6.9}$$

Since Δ and $\dot{\Delta}$ are analytic on $\mathbb{C}^* \times V_v$ and $\Delta^2(\lambda) - 1$ does not vanish on $\partial U_n \times V_v$, the right-hand side of the latter identity is analytic on V_v. Finally, if $v \in H_r^1$ then $\lambda_n^+, \lambda_n^- \in \mathbb{R}$ for any $n \in \mathbb{Z}$ (cf. Section 6.1). □

Lemma 6.8 *For each $n \in \mathbb{Z}$, $\tau_n = (\lambda_n^+ + \lambda_n^-)/2$, $\gamma_n^2 = (\lambda_n^+ - \lambda_n^-)^2$ define analytic functions on $\hat{W}$. Furthermore $(\gamma_n^2)_{n\ge0}$, $(n^4\gamma_{-n}^2)_{n\ge1} \in \ell^1$,*

$$\tau_n = n\pi + \ell_n^2, \quad \frac{1}{16\tau_{-n}} = n\pi + \ell_n^2$$

and

$$\partial\tau_n = (\ell_n^2 \cdot \partial_x(\cdot) + \ell_n^2, \quad \ell_n^2 \cdot P(\cdot)), \quad \partial(\gamma_n^2) = (\ell_n^2 \cdot \partial_x(\cdot), \quad \ell_n^2 \cdot P(\cdot))$$

uniformly in $0 \le x \le 1$ and locally uniformly on $\hat{W}$. We refer to Lemma 5.17 where the meaning of these asymptotic estimates are stated in detail.

Proof. Clearly, τ_n is analytic on $\hat{W}$ by Lemma 6.7. The same holds for γ_n^2 since

$$\gamma_n^2 = 2(\lambda_n^+)^2 + 2(\lambda_n^-)^2 - (\lambda_n^+ + \lambda_n^-)^2.$$

Further by Lemma 3.17, $(\gamma_n)_{n\geq 0} \in \ell^2$ and hence $(\gamma_n^2)_{n\geq 0} \in \ell^1$. Moreover $\gamma_{-n}(q,p) = \left(\frac{1}{16\lambda_n^-} - \frac{1}{16\lambda_n^+}\right)\Big|_{(-q,p)}$ and hence $(16\gamma_{-n}(q,p))^2 = \left(\frac{\gamma_n}{\lambda_n^-\lambda_n^+}\right)^2\Big|_{(-q,p)}$ implying that

$$(n^4\gamma_{-n}^2)_{n\geq 1} \in \ell^1. \tag{6.10}$$

Also by Lemma 3.17, one has $\tau_n = n\pi + \ell_n^2$ and hence for $n \geq 1$ sufficiently large

$$\frac{1}{16\tau_{-n}} = \frac{1}{16(\lambda_{-n}^- + \gamma_{-n}/2)} = \frac{1}{16\,\lambda_{-n}^-}(1+\gamma_{-n}/2\lambda_{-n}^-)^{-1} = (n\pi+\ell_n^2)(1+O\left(\frac{\gamma_{-n}}{n}\right)).$$

By (6.10) one then concludes that $\frac{1}{16\,\tau_{-n}} = n\pi + \ell_n^2$. To obtain the claimed estimates for the gradients note that for any given $v \in \hat{W}$ one has by (6.9)

$$2\tau_n = \frac{1}{2\pi\mathrm{i}}\int_{\partial U_n} \lambda\frac{\Delta(\lambda)\dot{\Delta}(\lambda)}{\Delta^2(\lambda)-1}\,\mathrm{d}\lambda$$

where $(U_n)_{n\in\mathbb{Z}}$, U_* is a set of isolating neighborhoods for v. Since

$$\frac{\Delta(\lambda)\dot{\Delta}(\lambda)}{\Delta^2(\lambda)-1} = \frac{1}{2}\partial_\lambda \log(\Delta^2(\lambda)-1)$$

for some appropriate branch of the logarithm and by the regularity of $\Delta(\lambda, v)$ in λ and v, the derivatives ∂ and ∂_λ commute, one sees that

$$2\partial\tau_n = \frac{1}{2\pi\mathrm{i}}\int_{\partial U_n} \lambda\partial_\lambda\left(\frac{\Delta(\lambda)\partial\Delta(\lambda)}{\Delta^2(\lambda)-1}\right)\lambda\,\mathrm{d}\lambda. \tag{6.11}$$

Integrating by parts then yields

$$2\partial\tau_n = -\frac{1}{2\pi\mathrm{i}}\int_{\partial U_n} \frac{\Delta(\lambda)\partial\Delta(\lambda)}{\Delta^2(\lambda)-1}\,\mathrm{d}\lambda.$$

On ∂U_n, $\Delta^2(\lambda)-1$ is bounded from below (Lemma 2.17), $\Delta(\lambda)$ is bounded from above (Lemma 3.14) and by Corollary 5.16

$$\partial\Delta(\lambda) = (\ell_n^2\cdot\partial_x(\cdot) + \ell_n^2, \quad \ell_n^2\cdot P(\cdot))$$

uniformly in $0 \leq x \leq 1$, $\lambda \in \partial U_n$, $n \in \mathbb{Z}$, and locally uniformly in v. Altogether this proves the claimed asymptotics for $\partial\tau_n$. For $\partial(\gamma_n^2)$ we proceed similarly. Since $\gamma_n^2 = 2(\lambda_n^+)^2 + 2(\lambda_n^-)^2 - (2\tau_n)^2$ one has in view of (6.9)

$$\begin{aligned}\partial\gamma_n^2 =& 4\frac{1}{2\pi\mathrm{i}}\int_{\partial U_n} \lambda^2\partial_\lambda\frac{\Delta(\lambda)\partial\Delta(\lambda)}{\Delta^2(\lambda)-1}\,\mathrm{d}\lambda - 8\tau_n\partial\tau_n\\ =& -8\frac{1}{2\pi\mathrm{i}}\int_{\partial U_n}(\lambda-\tau_n)\frac{\Delta(\lambda)\partial\Delta(\lambda)}{\Delta^2(\lambda)-1}\,\mathrm{d}\lambda.\end{aligned} \tag{6.12}$$

Arguing as above and using that $\lambda - \tau_n = O(1)$ on ∂U_n the claimed asymptotics follow. Going through the arguments of the proof one verifies the uniformity statements. □

To finish this section we show that on H_r^1, the nonvanishing of $\gamma_n = \lambda_n^+ - \lambda_n^-$ is generic for any $n \in \mathbb{Z}$. First we need to establish the following auxiliary result. Recall that $\grave{M} = \begin{pmatrix} \grave{m}_1 & \grave{m}_2 \\ \grave{m}_3 & \grave{m}_4 \end{pmatrix}$.

***Lemma* 6.9** *For any potential $v \in H_r^1$ the following statements are equivalent:*

(i) $\gamma_n = 0$.

(ii) $\grave{M}(\mu_n) = (-1)^n I$.

(iii) $\mu_n = \nu_n$ *and* $\grave{m}_1(\mu_n) = (-1)^n$.

Proof. If the n-th gap is collapsed then $\lambda_n^- = \mu_n = \nu_n = \lambda_n^+$ and hence $\grave{m}_2(\lambda_n^+) = 0$ and $\grave{m}_3(\lambda_n^+) = 0$. It follows that $\grave{M}(\lambda_n^+)$ is a diagonal matrix. Since $\mathrm{tr}\grave{M}(\lambda_n^+) = (-1)^n 2$ and $\det \grave{M}(\lambda_n^+) = 1$ it follows that $\grave{M}(\mu_n) = (-1)^n I$. Hence (i) implies (ii). Given (ii) one has $0 = \grave{m}_3(\mu_n) = \chi_N(\mu_n)$ and hence $\nu_n = \mu_n$. Thus (ii) implies (iii). Finally given (iii) we have that $\grave{m}_2(\mu_n) = \grave{m}_3(\mu_n) = 0$ and $\grave{m}_1(\mu_n) = (-1)^n$. Therefore the Wronskian identity reduces to

$$1 = \det \grave{M}(\mu_n) = (-1)^n \grave{m}_4(\mu_n)$$

and in turn $\grave{M}(\mu_n) = (-1)^n I$. This means that μ_n is a double periodic eigenvalue and by Lemma 6.1 one has $\mu_n = \lambda_n^+ = \lambda_n^-$. □

For any $n \in \mathbb{Z}$, introduce the set

$$Z_n := \{ \, v \in \hat{W} : \ \lambda_n^-(v) = \lambda_n^+(v) \, \}.$$

***Proposition* 6.10** *For any $n \in \mathbb{Z}$, the following holds:*

(i) Z_n is an analytic subvariety of $\hat{W}$.

(ii) $Z_n \cap H_r^1$ is contained in a codimension 1 submanifold, implying that $H_r^1 \setminus Z_n$ is dense in H_r^1.

Proof. (i) By Lemma 6.8, γ_n^2 is an analytic function on $\hat{W}$, which does not vanish identically.
(ii) First note that by Lemma 6.9, $Z_n \subset Y_n := \{ \ \grave{m}_1(\mu_n) = (-1)^n \ \}$. We claim that for an open neighborhood U of Z_n in H_r^1, $Y_n \cap U$ is a codimension 1 submanifold of H_r^1. First note that

$$\partial_p(\grave{m}_1(\mu_n)) = \dot{m}_1\Big|_{x=1} \partial_p \mu_n + \partial_p \grave{m}_1\Big|_{\mu_n}$$

and by (5.12) $P^{-1}(\partial_p \mu_n)\Big|_{x=0} = 0$. It then follows from (5.2) that for $v \in Z_n$,

$$P^{-1}(\partial_p \grave{m}_1(\mu_n))\Big|_{x=0} = P^{-1}(\partial_p \grave{m}_1)\Big|_{x=0, \lambda=\mu_n} = \frac{1}{4} \grave{m}_1\Big|_{\lambda=\mu_n} = \frac{1}{4}(-1)^n.$$

Hence there is an open neighborhood U of Z_n in H_r^1 such that $Y_n \cap U$ is a codimension 1 submanifold of H_r^1. □

6.3 Product representations

In this section we establish product representations of the characteristic functions $\chi_p(\lambda)$, $\chi_D(\lambda)$, and the function $\dot{\Delta}(\lambda)$, needed in the sequel. They require a different way of recording the roots of these functions which we now describe. For $v \in \hat{W}$ introduce

$$\lambda_{1,k}^+ := \begin{cases} \lambda_k^+ & k \geq 0 \\ -\lambda_{-k}^- & k \leq -1 \end{cases} \qquad \lambda_{1,k}^- := \begin{cases} \lambda_k^- & k \geq 0 \\ -\lambda_{-k}^+ & k \leq -1 \end{cases} \tag{6.13}$$

$$\lambda_{2,k}^+ := \begin{cases} \frac{1}{16\lambda_{-k}^-} & k \geq 0 \\ -\frac{1}{16\lambda_k^+} & k \leq -1 \end{cases} \qquad \lambda_{2,k}^- := \begin{cases} \frac{1}{16\,\lambda_{-k}^+} & k \geq 0 \\ -\frac{1}{16\,\lambda_k^-} & k \leq -1 \end{cases} \tag{6.14}$$

We note that for any $k \geq 1$ and $j = 1, 2$

$$\lambda_{j,-k}^+ = -\lambda_{j,k}^-, \quad \lambda_{j,-k}^- = -\lambda_{j,k}^+ \tag{6.15}$$

and

$$\lambda_{1,0}^+ = \frac{1}{16\,\lambda_{2,0}^-}, \quad \lambda_{1,0}^- = \frac{1}{16\,\lambda_{2,0}^+}. \tag{6.16}$$

According to Lemma 3.17,

$$\lambda_{j,k}^+, \lambda_{j,k}^- = k\pi + \ell_k^2, \quad k \in \mathbb{Z}, \ j = 1, 2 \tag{6.17}$$

and by Lemma 6.6

$$\lambda_{2,k}^+(q,p) = \lambda_{1,k}^+(-q,p), \quad \lambda_{2,k}^-(q,p) = \lambda_{1,k}^-(-q,p), \ k \in \mathbb{Z}. \tag{6.18}$$

Furthermore if v is real valued (hence in H^1_r)

$$\cdots < \lambda^-_{j,k} \leq \lambda^+_{j,k} < \lambda^-_{j,k+1} \leq \lambda^+_{j,k+1} < \cdots \tag{6.19}$$

Similarly we define

$$\mu_{1,k} := \begin{cases} \mu_k & k \geq 0 \\ -\mu_{-k} & k \leq -1 \end{cases} \qquad \mu_{2,k} := \begin{cases} \frac{1}{16\mu_{-k}} & k \geq 0 \\ -\frac{1}{16\mu_k} & k \leq -1 \end{cases} \tag{6.20}$$

and

$$\dot\lambda_{1,k} := \begin{cases} \dot\lambda_k & k \geq 0 \\ -\dot\lambda_{-k} & k \leq -1 \end{cases} \qquad \dot\lambda_{2,k} := \begin{cases} \frac{1}{16\dot\lambda_{-k}} & k \geq 0 \\ -\frac{1}{16\dot\lambda_k} & k \leq -1 \end{cases}. \tag{6.21}$$

Before starting the product representation of $\chi_p(\lambda)$, $\chi_D(\lambda)$, and $\dot\Delta(\lambda)$, let us discuss the envisioned type of representations in general terms. According to Lemma B.1 for any given sequences $(\sigma_{1,n})_n$, $(\sigma_{2,n})_n$ in the space

$$\ell^* := \{ \lambda_n = n\pi + \ell^2_n \; : \; \lambda \in \mathbb{C}^*, \; n \in \mathbb{Z} \}$$

the infinite products

$$f_1(\lambda) := \prod_{n\in\mathbb{Z}} \frac{\sigma_{1,n} - \lambda}{\pi_n}, \quad f_2(\lambda) := \prod_{n\in\mathbb{Z}} \frac{\sigma_{2,n} + \frac{1}{16\lambda}}{\pi_n} \tag{6.22}$$

define analytic functions on $\mathbb{C}^*$ with roots $\sigma_{1,n}$, $n \in \mathbb{Z}$, and respectively, $(-16\sigma_{2,n})^{-1}$, $n \in \mathbb{Z}$. Note that in addition, f_1 is analytic at 0, f_2 is analytic at ∞ and

$$f_1(0) = \prod_{n\in\mathbb{Z}} \frac{\sigma_{1,n}}{\pi_n}, \quad f_2(\infty) = \prod_{n\in\mathbb{Z}} \frac{\sigma_{2,n}}{\pi_n} \tag{6.23}$$

are well defined numbers in $\mathbb{C}^*$. Furthermore, by Lemma B.5, one sees that uniformly for λ in $\partial B_N = \{ \lambda \in \mathbb{C} : \; |\lambda| = N\pi + \pi/2 \}$

$$f_1(\lambda) = -(1 + o(1)) \sin(\lambda), \quad f_2(\lambda) = f_2(\infty) + O(\frac{1}{N}), \quad \text{as } N \to \infty \tag{6.24}$$

and uniformly for λ in

$$\partial B_{-N} = \{ \lambda \in \mathbb{C} : \; |16\lambda| = \frac{1}{N\pi + \pi/2} \}$$

$$f_1(\lambda) = f_1(0) + O(\frac{1}{N}), \quad f_2(\lambda) = (1 + o(1)) \sin(\frac{1}{16\lambda}), \quad \text{as } N \to \infty. \tag{6.25}$$

Let us first consider $\dot\Delta(\lambda) \equiv \dot\Delta(\lambda, v)$ for $v \in \hat W$. Since by Lemma 3.15, $\dot\lambda_{j,n} = n\pi + \ell^2_n$ for $j = 1, 2$, the infinite products

$$\dot\Delta_1(\lambda) := \prod_{n\in\mathbb{Z}} \frac{\dot\lambda_{1,n} - \lambda}{\pi_n}, \quad \dot\Delta_2(\lambda) := \prod_{n\in\mathbb{Z}} \frac{\dot\lambda_{2,n} + \frac{1}{16\lambda}}{\pi_n} \tag{6.26}$$

are well defined analytic functions on $\mathbb{C}^*$ by the considerations above.

***Lemma* 6.11** *On $\hat W$, $\dot\Delta(\lambda)$ admits the product representation for $\lambda \in \mathbb{C}^*$*

$$\dot\Delta(\lambda) = c_{\dot\Delta}(1 - \frac{\dot\lambda_*}{\lambda})(1 + \frac{\dot\lambda_*}{\lambda})\dot\Delta_1(\lambda)\dot\Delta_2(\lambda), \quad c_{\dot\Delta} := \frac{1}{\dot\Delta_2(\infty)}. \tag{6.27}$$

Furthermore, $\dot\Delta_2(\infty) = -16\dot\lambda_*^2 \dot\Delta_1(0)$ or in more detail

$$-\dot\lambda_*^2(16\dot\lambda_0)^2 \prod_{n\geq 1} (\dot\lambda_n 16\dot\lambda_{-n})^2 = 1. \tag{6.28}$$

Proof. Let $F(\lambda) := \dot\Delta_1(\lambda)\dot\Delta_2(\lambda)$. Then $F(\lambda)$ is analytic on $\mathbb{C}^*$. By (6.24), uniformly for $\lambda \in \partial B_N$

$$F(\lambda) = -(1 + o(1))\dot\Delta_2(\infty) \sin(\lambda) \quad \text{as } N \to \infty \tag{6.29}$$

and by (6.25), uniformly for $\lambda \in \partial B_{-N}$

$$F(\lambda) = (1 + o(1))\dot{\Delta}_1(0)\sin(\frac{1}{16\lambda}) \quad \text{as } N \to \infty. \tag{6.30}$$

Hence $(1 - \frac{\dot{\lambda}_*}{\lambda})(1 + \frac{\dot{\lambda}_*}{\lambda})F(\lambda)$ is a holomorphic function on $\mathbb{C}^*$ with the same roots as $\dot{\Delta}(\lambda)$ and with the property that the quotient $G(\lambda) = (1 - \frac{\dot{\lambda}_*}{\lambda})(1 + \frac{\dot{\lambda}_*}{\lambda})F(\lambda)/\dot{\Delta}(\lambda)$ defines a holomorphic function on $\mathbb{C}^*$. By the asymptotics of $\dot{\Delta}$ of Lemma 2.17 and (6.29), one has uniformly in $\lambda \in \partial B_N$

$$G(\lambda) = \dot{\Delta}_2(\infty)(1 + o(1)) \quad \text{as } N \to \infty. \tag{6.31}$$

To obtain the asymptotics of G for $\lambda \in \partial B_{-N}$ as $N \to \infty$ note that by Lemma 2.14(ii), $\dot{\Delta}(\lambda, q, p) = \dot{\Delta}(-\frac{1}{16\lambda}, -q, p) \cdot \frac{1}{16\lambda^2}$ which we rewrite with $\lambda = -\frac{1}{16\mu}$ as

$$\dot{\Delta}(-\frac{1}{16\mu}, q, p) = \dot{\Delta}(\mu, -q, p)16\mu^2.$$

Since $\lambda \in \partial B_{-N}$ iff $\mu = -\frac{1}{16\lambda} \in \partial B_N$ it then follow from (6.30) that uniformly for $\lambda \in \partial B_{-N}$

$$G(\lambda) = G(-\frac{1}{16\mu}) = \frac{1 - (16\mu\dot{\lambda}_*)^2}{16\mu^2}\dot{\Delta}_1(0)(1 + o(1)) \quad \text{as } N \to \infty.$$

Hence G is bounded on ∂B_{-N} as $N \to \infty$. This together with (6.31) allows to apply Lemma C.1 (Liouville's theorem) yielding that G is constant. One concludes that (6.27) holds and $\dot{\Delta}_2(\infty) = -16(\dot{\lambda}_*)^2\dot{\Delta}_1(0)$ or, with $\dot{\Delta}_2(\infty) = \prod_{n\in\mathbb{Z}} \frac{\dot{\lambda}_{2,n}}{\pi_n}$, $\dot{\Delta}_1(0) = \prod_{n\in\mathbb{Z}} \frac{\dot{\lambda}_{1,n}}{\pi_n}$,

$$1 = -16\dot{\lambda}_*^2 \prod_{n\in\mathbb{Z}} \frac{\dot{\lambda}_{1,n}}{\dot{\lambda}_{2,n}}.$$

Taking into account identities of the type (6.15)-(6.16), it then follows that

$$-\dot{\lambda}_*^2(16\dot{\lambda}_0)^2 \prod_{n\geq 1}(\dot{\lambda}_n 16\dot{\lambda}_{-n})^2 = 1$$

as claimed. □

Remark 6.12. Note that for $v = 0$, $\dot{\lambda}_* = \mathrm{i}/4$ and by Lemma 6.6(i) $\frac{1}{16\dot{\lambda}_{-n}} = \dot{\lambda}_n$ for any $n \in \mathbb{Z}$.

In the same way one can prove a corresponding product representation for $\chi_D(\lambda) \equiv \chi_D(\lambda, v)$ with $v \in \hat{W}$. Since by Lemma 3.16, $\mu_{j,n} = n\pi + \ell_n^2$ for $j = 1, 2$, the infinite products

$$\chi_{D,1}(\lambda) := \prod_{n\in\mathbb{Z}} \frac{\mu_{1,n} - \lambda}{\pi_n}, \quad \chi_{D,2}(\lambda) := \prod_{n\in\mathbb{Z}} \frac{\mu_{2,n} + \frac{1}{16\lambda}}{\pi_n}$$

are well defined analytic functions on $\mathbb{C}^*$.

***Lemma* 6.13** *On $\hat{W}$, χ_D admits the product representation for $\lambda \in \mathbb{C}^*$*

$$\chi_D(\lambda, v) = -c_D\chi_{D,1}(\lambda)\chi_{D,2}(\lambda), \quad c_D = \frac{1}{\chi_{D,2}(\infty)}. \tag{6.32}$$

Furthermore $\chi_{D,2}(\infty) = -e^{q(0)}\chi_{D,1}(0)$ or in more detail,

$$e^{q(0)}16\mu_0^2 \prod_{n\geq 1}(\mu_n 16\mu_{-n})^2 = 1$$

and

$$\chi_{D,2}(-\frac{1}{16\lambda}, q, p) = \chi_{D,1}(\lambda, -q, p).$$

Proof. Let $F(\lambda) = \chi_{D,1}(\lambda)\chi_{D,2}(\lambda)$. Then F is analytic on $\mathbb{C}^*$. Since F and $\chi_D(\lambda)$ have the same roots, $G(\lambda) := F(\lambda)/\chi_D(\lambda)$ is well defined and analytic on $\mathbb{C}^*$. Arguing as in the proof of Lemma 6.11 one sees that uniformly for $\lambda \in \partial B_N$

$$F(\lambda) = -(1 + o(1))\chi_{D,2}(\infty)\sin(\lambda) \quad \text{as } N \to \infty \tag{6.33}$$

and uniformly for $\lambda \in \partial B_{-N}$

$$F(\lambda) = (1 + o(1))\chi_{D,1}(0)\sin(\frac{1}{16\lambda}). \tag{6.34}$$

By Lemma 3.2(iii), uniformly for $\lambda \in \partial B_N$,

$$\chi_D(\lambda, v) = (1 + o(1))\chi_D(\lambda, 0) \quad \text{as } N \to \infty.$$

Since for $v = 0$, $\chi_D(\lambda, 0) = \sin(\omega(\lambda))$ (Theorem 3.1) and by (3.19), $\dot{\Delta}(\lambda, 0) = -(1 + \frac{1}{!6\lambda^2})\sin(\omega(\lambda))$ and $\dot{\lambda}_* = \mathrm{i}/4$, one has

$$\chi_D(\lambda, 0) = -c_{\dot{\Delta}}\dot{\Delta}_1(\lambda, 0)\dot{\Delta}_2(\lambda, 0).$$

Hence by (6.29) uniformly for $\lambda \in \partial B_N$,

$$\chi_D(\lambda, v) = (1 + o(1))\chi_D(\lambda, 0) = \sin(\lambda)(1 + o(1)) \quad \text{as } N \to \infty. \tag{6.35}$$

This combined with (6.33) then shows that uniformly for $\lambda \in \partial B_N$

$$G(\lambda) = -\chi_{D,2}(\infty)(1 + o(1)) \quad \text{as } N \to \infty.$$

To obtain the asymptotics of G for $\lambda \in \partial B_{-N}$ as $N \to \infty$, note that by Lemma 3.2,

$$\chi_D(\lambda, q, p) = e^{-q(0)}\chi_D(-\frac{1}{16\lambda}, -q, p).$$

Arguing as in the proof of Lemma 6.11 one then concludes from (6.34) that uniformly for $\lambda \in \partial B_{-N}$

$$G(\lambda) = -e^{q(0)}\chi_{D,1}(0)(1 + o(1)).$$

Hence by Lemma C.1, $G(\lambda)$ is constant, implying that

$$\chi_D(\lambda) = -\frac{1}{\chi_{D,2}(\infty)}\chi_{D,1}(\lambda)\chi_{D,2}(\lambda)$$

as well as $\chi_{D,2}(\infty) = e^{q(0)}\chi_{D,1}(0)$ or in more detail,

$$1 = e^{q(0)}\frac{\chi_{D,1}(0)}{\chi_{D,2}(\infty)} = e^{q(0)}\prod_{n\in\mathbb{Z}}\frac{\mu_{1,n}}{\mu_{2,n}}.$$

Taking into account the identities of the type (6.15)-(6.16) one obtains the claimed identity

$$1 = e^{q(0)}16\mu_0^2\prod_{n\geq 1}(\mu_n 16\mu_{-n})^2.$$

Since by Lemma 6.5

$$\mu_{2,n}(q, p) = \mu_{1,n}(-q, p) \quad \forall\, n \in \mathbb{Z},$$

one has

$$\chi_{D,2}(-\frac{1}{16\lambda}, q, p) = \prod_{n\in\mathbb{Z}}\frac{\mu_{2,n}(q, p) - \lambda}{\pi_n} = \chi_{D,1}(\lambda, -q, p).$$

□

Finally, we discuss the product representation of $\chi_p(\lambda, v) = \Delta^2(\lambda, v) - 1$ for $v \in \hat{W}$. Since by Lemma 3.17, $\lambda^{\pm}_{j,n} = n\pi + \ell^2_n$ for $j = 1, 2$, the infinite products

$$\chi_{p,1}(\lambda) := \prod_{n\in\mathbb{Z}}\frac{(\lambda^+_{n,1} - \lambda)(\lambda^-_{1,n} - \lambda)}{\pi_n^2}, \quad \chi_{p,2}(\lambda) := \prod_{n\in\mathbb{Z}}\frac{(\lambda^+_{n,2} + \frac{1}{16\,\lambda})(\lambda^-_{2,n} + \frac{1}{16\,\lambda})}{\pi_n^2},$$

are well defined analytic functions on $\mathbb{C}^*$. Note that

$$\chi_{p,1}(0) = \prod_{n\in\mathbb{Z}} \frac{\lambda_{1,n}^+ \lambda_{1,n}^-}{\pi_n^2} = \lambda_0^+ \lambda_0^- \left(\prod_{n\geq 1} \frac{\lambda_n^+ \lambda_n^-}{\pi_n^2} \right)^2 \tag{6.36}$$

and

$$\chi_{p,2}(\infty) = \prod_{n\in\mathbb{Z}} \frac{\lambda_{2,n}^+ \lambda_{2,n}^-}{\pi_n^2} = (16\lambda_0^+ 16\lambda_0^-)^{-1} \left(\prod_{n\geq 1} \frac{(16\lambda_{-n}^+ 16\lambda_{-n}^-)^{-1}}{\pi_n^2} \right)^2$$

***Lemma* 6.14** *On $\hat{W}$, χ_p admits the product representation for $\lambda \in \mathbb{C}^*$*

$$\chi_p(\lambda) = -c_p \chi_{p,1}(\lambda)\chi_{p,2}(\lambda), \quad c_p = \frac{1}{\chi_{p,1}(0)}. \tag{6.37}$$

Furthermore,

$$\chi_{p,2}(\infty) = \chi_{p,1}(0) \tag{6.38}$$

or in more detail,

$$(16\lambda_0^+ \lambda_0^-)^2 \prod_{n\geq 1} (\lambda_n^+ 16\lambda_{-n}^+)^2 (\lambda_n^- 16\lambda_{-n}^-)^2 = 1, \tag{6.39}$$

and

$$\chi_{p,2}\big(-\frac{1}{16\lambda}, q, p\big) = \chi_{p,1}(\lambda, -q, p). \tag{6.40}$$

As a consequence, $\chi_{p,1}(0, -q, p) = \chi_{p,1}(0, q, p)$.

Proof. Let $F(\lambda) = \chi_{p,1}(\lambda)\chi_{p,2}(\lambda)$. Then $F(\lambda)$ is analytic on $\mathbb{C}^*$. Since $F(\lambda)$ and $\chi_p(\lambda)$ have the same roots, $G(\lambda) := F(\lambda)/\chi_p(\lambda)$ is well defined and analytic on $\mathbb{C}^*$. Arguing as in the proof of Lemma 6.11, one sees that uniformly for $\lambda \in \partial B_N$

$$F(\lambda) = (1 + o(1))\chi_{p,2}(\infty)\sin^2(\lambda) \quad \text{as } N \to \infty \tag{6.41}$$

and uniformly for $\lambda \in \partial B_{-N}$

$$F(\lambda) = (1 + o(1))\chi_{p,1}(0)\sin^2\Big(\frac{1}{16\lambda}\Big) \quad \text{as } N \to \infty. \tag{6.42}$$

Since by Lemma 2.17, uniformly for $\lambda \in \partial B_N$

$$\chi_p(\lambda) = -(1 + o(1))\sin^2(\lambda)$$

it then follows from (6.41) that uniformly for $\lambda \in \partial B_N$

$$G(\lambda) = -(1 + o(1))\chi_{p,2}(\infty) \quad \text{as } N \to \infty. \tag{6.43}$$

To obtain the asymptotics of G on ∂B_{-N} note that by Lemma 2.14(i),(ii)

$$\chi_p\Big(-\frac{1}{16\mu}, q, p\Big) = \chi_p(\mu, -q, p).$$

When combined with (6.42) and (6.43) one then concludes that uniformly for $\lambda \in \partial B_{-N}$

$$G(\lambda) = -(1 + o(1))\chi_{p,1}(0) \quad \text{as } N \to \infty.$$

Hence by Lemma C.1, G is constant and therefore

$$\chi_p(\lambda) = -\frac{1}{\chi_{p,2}(\infty)}\chi_{p,1}(\lambda)\chi_{p,2}(\lambda)$$

and $\chi_{p,2}(\infty) = \chi_{p,1}(0)$ which can be expressed as

$$1 = (16\lambda_0^+ \lambda_0^-)^2 \prod_{n\geq 1} (\lambda_n^+ 16\lambda_{-n}^+)^2 (\lambda_n^- 16\lambda_{-n}^-)^2.$$

Furthermore, since by Lemma 6.6, $(16\lambda^{\pm}_{-n}(q,p))^{-1} = \lambda^{\mp}_{n}(-q,p)$ for any $n \in \mathbb{Z}$, implying that $\lambda^{\pm}_{2,n}(q,p) = \lambda^{\pm}_{1,n}(-q,p)$ for any $n \in \mathbb{Z}$, and since $\lambda^{+}_{j,n} = -\lambda^{-}_{j,-n}$ for any $n \neq 0$, $j = 1, 2$ one sees that

$$\chi_{p,2}(-\frac{1}{16\mu}, q, p) = \prod_{n\in\mathbb{Z}} \frac{(\lambda^{+}_{2,n}(q,p) - \mu)(\lambda^{-}_{2,n}(q,p) - \mu)}{\pi_n^2} = \chi_{p,1}(\mu, -q, p),$$

proving (6.40). For $\mu = 0$, one then gets $\chi_{p,1}(0, -q, p) = \chi_{p,2}(\infty, q, p)$ which equals $\chi_{p,1}(0, q, p)$ by (6.38). □

To finish this section we prove asymptotics for the sequences $(\tau_n - \dot{\lambda}_n)_n$ and asymptotics for $(\dot{\lambda}_n)_n$ as $n \to \infty$, refining the ones of Lemma 3.15. Recall that by (6.8), $\tau_n = (\lambda^+_n + \lambda^-_n)/2$. For $n \geq 0$, let $\Delta_n(\lambda)$ be defined by

$$\chi_p(\lambda) = \Delta^2(\lambda) - 1 = \Delta_n(\lambda)(\lambda^+_n - \lambda)(\lambda^-_n - \lambda).$$

By Lemma 6.13, $\Delta_n(\lambda)$ admits the product representation

$$\Delta_n(\lambda) = -c_p \chi_{p,2}(\lambda) \frac{\chi_{p,1}(\lambda)}{(\lambda^+_n - \lambda)(\lambda^-_n - \lambda)}. \tag{6.44}$$

***Lemma* 6.15** *On $\hat{W}$ for any $n \geq 0$,*

$$2(\tau_n - \dot{\lambda}_n)\Delta_n(\dot{\lambda}_n) = \left((\tau_n - \dot{\lambda}_n)^2 - \gamma_n^2/4\right) \dot{\Delta}_n(\dot{\lambda}_n). \tag{6.45}$$

Furthermore, locally uniformly on $\hat{W}$,

$$\dot{\lambda}_n = \tau_n + \gamma_n^2 \ell_n^2 \quad \textit{as } n \to \infty. \tag{6.46}$$

Proof. Since $\dot{\Delta}(\dot{\lambda}_n)$, one has

$$0 = \partial_\lambda(\Delta^2(\lambda) - 1)\big|_{\lambda=\dot{\lambda}_n} = -(\lambda^+_n - 2\dot{\lambda}_n + \lambda^-_n)\Delta_n(\dot{\lambda}_n) + (\lambda^+_n - \dot{\lambda}_n)(\lambda^-_n - \dot{\lambda}_n)\dot{\Delta}_n(\dot{\lambda}_n).$$

The identity $(\lambda^+_n - \dot{\lambda}_n)(\lambda^-_n - \dot{\lambda}_n) = (\tau_n - \dot{\lambda}_n)^2 - \gamma_n^2/4$ then yields (6.45). To prove the asymptotics (6.46), note that by Lemma B.4

$$\frac{\chi_{p,1}(\lambda)}{(\lambda^+_n - \lambda)(\lambda^-_n - \lambda)} = \left(\frac{\sin(\lambda - n\pi)}{\lambda - n\pi}\right)^2 (1 + \ell_n^2)$$

uniformly for $\lambda \in D_n$. On the other hand,

$$\chi_{p,2}(\lambda) = \chi_{p,2}(\infty) + O(\frac{1}{n}) = c_p^{-1} + O(\frac{1}{n})$$

uniformly for $\lambda \in D_n$. Altogether it then follows that

$$\Delta_n(\lambda) = -\left(\frac{\sin(\lambda - n\pi)}{\lambda - n\pi}\right)^2 + \ell_n^2$$

uniformly for $\lambda \in D_n$ as $n \to \infty$. Hence by Cauchy's estimate

$$\partial_\lambda \left(\Delta_n(\lambda) + \left(\frac{\sin(\lambda - n\pi)}{\lambda - n\pi}\right)^2\right) = \ell_n^2$$

uniformly for $\lambda \in D_n$. Using that $\frac{\sin(\lambda - n\pi)}{\lambda - n\pi} = \int_0^1 \cos(t(\lambda - n\pi))\,\mathrm{d}t$ one sees that

$$\partial_\lambda \left(\frac{\sin(\lambda - n\pi)}{\lambda - n\pi}\right) = -\int_0^1 t\sin(t(\lambda - n\pi))\,\mathrm{d}t.$$

Since $\dot{\lambda}_n = n\pi + \ell_n^2$, this implies that

$$\partial_\lambda \left(\frac{\sin(\lambda - n\pi)}{\lambda - n\pi}\right)^2 \Big|_{\lambda=\dot{\lambda}_n} = \ell_n^2.$$

Altogether we have shown that

$$\dot{\Delta}_n(\dot{\lambda}_n) = \ell_n^2, \quad \Delta_n(\dot{\lambda}_n) = -1 + \ell_n^2. \tag{6.47}$$

Hence (6.45) implies that

$$(\tau_n - \dot{\lambda}_n)(1 + \ell_n^2 + (\tau_n - \dot{\lambda}_n)\ell_n^2) = \gamma_n^2 \ell_n^2.$$

Since by Lemma 3.15 and Lemma 3.17, $\tau_n - \dot{\lambda}_n = \ell_n^2$ there exists $N \geq 1$ so that $(1+\ell_n^2+(\tau_n-\dot{\lambda}_n)\ell_n^2) \geq \frac{1}{2}$ for any $n \geq N$ and (6.46) follows. Going through the arguments of the proof one verifies that (6.46) holds locally uniformly on $\hat{W}$. □

6.4 Standard roots

In this section we introduce various branches of square roots which we need later for constructing actions and angles. Customarily, we denote by $\sqrt[+]{\lambda}$ the principal branch of th square root, defined for $\lambda \in \mathbb{C} \setminus (-\infty, 0]$. For $v \in \hat{W}$, $n \in \mathbb{Z}$, $j \in \{1, 2\}$ set

$$\gamma_{j,n} := \lambda_{j,n}^+ - \lambda_{j,n}^-, \quad \tau_{j,n} := (\lambda_{j,n}^+ + \lambda_{j,n}^-)/2.$$

Note that since $\lambda_{j,-n}^+ = -\lambda_{j,n}^-$, $\lambda_{j,-n}^- = -\lambda_{j,n}^+$ for any $n \geq 1$ one has

$$\gamma_{j,-n} = \gamma_{j,n}, \quad \tau_{j,-n} = -\tau_{j,n} \ \ \forall\, n \geq 1, \ j = 1, 2.$$

Furthermore define

$$G_{1,n} := [\lambda_n^-, \lambda_n^+] \ (n \geq 0), \quad G_{1,-n} := -G_{1,n} \ (n \geq 1)$$
$$G_{2,n} := [-\lambda_{-n}^+, -\lambda_{-n}^-] \ (n \geq 0), \quad G_{2,-n} := -G_{2,n} \ (n \geq 1).$$

Note that the $G_{j,n}$ have the following symmetry: with $\mathcal{S}_{rec}$ denoting the map

$$\mathcal{S}_{rec} : \mathbb{C}^* \to \mathbb{C}^*, \lambda \mapsto -\frac{1}{16\lambda}$$

one has for any $v = (q, p) \in \hat{W}$ and $n \in \mathbb{Z}$

$$\mathcal{S}_{rec}(G_{1,n}(q, p)) = G_{2,n}(-q, p), \quad \mathcal{S}_{rec}(G_{2,n}(q, p)) = G_{1,n}(-q, p).$$

Finally let

$$U_{1,n} := U_n \ (n \geq 0), \quad U_{1,-n} := -U_n \ (n \geq 1),$$
$$U_{2,n} := -U_{-n} \ (n \geq 0), \quad U_{2,-n} := U_{-n} \ (n \geq 1),$$

where U_n, $n \in \mathbb{Z}$, are isolating neighborhoods for $v \in \hat{W}$. Without loss of generality we can (and in the sequel will) assume that for any $v = (q, p) \in \hat{W}$, $n \in \mathbb{Z}$

$$\mathcal{S}_{rec}(U_{1,n}(q, p)) = U_{2,n}(-q, p), \quad \mathcal{S}_{rec}(U_{2,n}(q, p)) = U_{1,n}(-q, p).$$

***Definition* 6.16** *For any $n \in \mathbb{Z}$, $v \in \hat{W}$, the standard root $w_{1,n}(\lambda) \equiv w_{1,n}(\lambda, v)$, also referred to as s-root, is defined by*

$$w_{1,n}(\lambda) \equiv w_{1,n}(\lambda, v) := \sqrt[s]{(\lambda_{1,n}^+ - \lambda)(\lambda_{1,n}^- - \lambda)}, \quad \lambda \notin G_{1,n}$$

determined by setting for $\lambda \in \mathbb{C}$ with $\left|\gamma_{1,n}^2/4(\tau_n - \lambda)^2\right| < 1$,

$$w_{1,n}(\lambda) = (\tau_{1,n} - \lambda) \sqrt[+]{1 - \gamma_{1,n}^2/4(\tau_{1,n} - \lambda)^2}. \tag{6.48}$$

Note that for $n \geq 1$, since $\gamma_{1,n}^2/4(\tau_{1,n} + \lambda)^2 = \gamma_{1,-n}^2/4(\tau_{1,-n} - \lambda)^2$, (6.48) implies that

$$w_{1,-n}(\lambda) = -(\tau_{1,n} + \lambda) \sqrt[+]{1 - \gamma_{1,n}^2/4(\tau_{1,n} + \lambda)^2} = -w_{1,n}(-\lambda). \tag{6.49}$$

for any $\lambda \in \mathbb{C}$ with $|\gamma_{1,n}^2/4(\tau_{1,n} + \lambda)^2| < 1$.

Lemma 6.17 *For any $v \in \hat{W}$ and $n \in \mathbb{Z}$, the standard root $w_{1,n}(\lambda)$ is analytic on $\mathbb{C} \setminus G_{1,n}$. In case $\gamma_{1,n} = 0$, $w_{1,n}(\lambda) = \tau_{1,n} - \lambda$. Furthermore, for any $v_0 \in \hat{W}$, $w_{1,n}(\lambda, v)$ is analytic in on $(\mathbb{C}^* \setminus U_{1,n}) \times V_{v_0}$ where V_{v_0} is the open neighborhood of Lemma 6.4 and U_n, $n \in \mathbb{Z}$ are isolating neighborhoods for V_{v_0}. Moreover, there exists a constant $c \geq 1$, such that for any $n, m \in \mathbb{Z}$, with $m \neq n$,*

$$c^{-1}|m-n| \leq |w_{1,n}(\lambda, v)| \leq c|m-n|, \quad (\lambda, v) \in U_{1,m} \times V_{v_0}.$$

Finally, for $n \in \mathbb{Z}$

$$\frac{1}{2\pi \mathrm{i}} \int_{\Gamma_{1,m}} \frac{\mathrm{d}\lambda}{w_n(\lambda, v)} = -\delta_{mn}$$

where $\Gamma_{1,m} := \Gamma_m$, $m \geq 0$, and $\Gamma_{1,-m} := (\Gamma_m)^-$, $m \geq 1$.

Proof. The claimed result can be proved in a straight forward way using the asymptotics of the periodic eigenvalues of $Q(v)$ of Lemma 3.17. □

Next we want to define the canonical root $\sqrt[c]{\chi_{1,p}(\lambda)}$ in terms of the roots $w_{1,n}(\lambda)$. To this end we need the following lemma.

Lemma 6.18 *(i) Let $v_0 \in \hat{W}$ be given. For any $v \in V_{v_0}$ and $n \geq 0$,*

$$f_{1,n}(\lambda, v) := \frac{1}{\pi_n} \prod_{m \neq n} \frac{w_{1,m}(\lambda, v)}{\pi_m} \tag{6.50}$$

defines a function which is analytic in λ on $\mathbb{C} \setminus \bigcup_{m \neq n} G_{1,m}$ and analytic in both variables on $\mathbb{C} \setminus \left(\bigcup_{m \neq n} U_{1,m}\right) \times V_{v_0}$. Moreover, $f_{1,n}$ does not vanish on these domains and in case $\gamma_{1,m} = 0$, it extends analytically in λ to $\lambda = \tau_{1,m}$.

(ii) For any $v \in H^1_r$ and $n \in \mathbb{Z}$,

$$(-1)^n f_{1,n}(\lambda, v) > 0 \quad \forall\ \lambda^+_{1,n-1} < \lambda < \lambda^-_{1,n+1}.$$

Proof. Item (i) follows from Lemma 6.17. Concerning (ii) note that for $v \in H^1_r$, $\lambda^\pm_{1,n} \in \mathbb{R}^*$ and by (6.48), $w_{1,m}(\lambda)$ is real valued and positive for $\lambda < \lambda^-_{1,m}$ and real and negative for $\lambda > \lambda^+_{1,m}$. Hence for $\lambda^+_{1,n-1} < \lambda < \lambda^-_{1,n+1}$, with $n \geq 0$

$$\prod_{m > n} \frac{w_{1,m}(\lambda, v) w_{1,-m}(\lambda, v)}{\pi_m \pi_{-m}} > 0, \qquad \prod_{0 \leq m < n} \frac{w_{1,m}(\lambda, v)}{\pi_m} > 0$$

and

$$(-1)^n \prod_{-n \leq m < 0} \frac{w_{1,m}(\lambda, v)}{\pi_m} > 0.$$

In the case where $n \leq -1$, one argues similarly. □

By the definition of $\chi_{p,1}(\lambda)$ and $f_{1,n}$, $n \in \mathbb{Z}$, one has on $\mathbb{C}$

$$\chi_{p,1}(\lambda) = w_n^2(\lambda) f_{1,n}^2(\lambda).$$

The canonical root $\sqrt[c]{\chi_1}$ of $\chi_1(\lambda) \equiv \chi_{p,1}(\lambda)$ is then defined on $\mathbb{C} \setminus \bigcup_{n \in \mathbb{Z}} G_{1,m}$ by

$$\sqrt[c]{\chi_1(\lambda)} := w_{1,n}(\lambda) f_{1,n}(\lambda). \tag{6.51}$$

Lemma 6.19 *On $\hat{W}$, $\sqrt[c]{\chi_1(\lambda, v)}$ is well defined by (6.51) on $\mathbb{C} \setminus \bigcup_{m \in \mathbb{Z}} G_{1,m}(v)$ and for any $v_0 \in \hat{W}$ analytic in (λ, v) on $(\mathbb{C} \setminus \bigcup_{m \in \mathbb{Z}} U_{1,m}) \times V_{v_0}$. Moreover, $\sqrt[c]{\chi_1(\lambda, v)}$ does not vanish on these domains,*

$$\sqrt[c]{\chi_1(0, q, p)} = \sqrt[c]{\chi_1(0, -q, p)}, \tag{6.52}$$

and for $\lambda \in \mathbb{C} \setminus \bigcup_{m \in \mathbb{Z}} U_{1,m}$ (and hence $-\lambda \in \mathbb{C} \setminus \bigcup_{m \in \mathbb{Z}} U_{1,m}$)

$$\sqrt[c]{\chi_1(-\lambda, v)} = \sqrt[c]{\chi_1(\lambda, v)} \frac{w_{1,0}(-\lambda, v)}{w_{1,0}(\lambda, v)}. \tag{6.53}$$

In case $\gamma_{1,m}(v) = 0$ for some $m \in \mathbb{Z}$, $\sqrt[c]{\chi_1(\lambda, v)}$ extends analytically to $\lambda = \tau_{1,m}(v)$.

Proof. In view of Lemma 6.18, it remains to prove identity (6.52) and (6.53). Since by Lemma 6.14, $\chi_1(0,q,p) = \chi_1(0,-q,p)$, the claimed identity (6.52) is true up to a sign. Since $\chi_1(0,v) \neq 0$ for any $v \in \hat{W}$ and (6.52) clearly holds for $v = 0$, (6.52) follows by continuity. Concerning (6.53) note that by (6.49), for any $n \neq 0$, $w_{1,-n}(\lambda)/\pi_{-n} = w_{1,n}(-\lambda)/\pi_n$. This then implies (6.53). □

Lemma 6.20 *On H^1_r, the following holds for any $n \in \mathbb{Z}$:*

(i) For any $\lambda^+_{1,n-1} < \lambda < \lambda^-_{1,n}$, $(-1)^n \sqrt[c]{\chi_1(\lambda)} > 0$.

(ii) For any $\lambda^-_{1,n} \leq \lambda \leq \lambda^+_{1,n}$, the limits of $\sqrt[c]{\chi_1(\lambda + \mathrm{i}\epsilon)}$ and $\sqrt[c]{\chi_1(\lambda - \mathrm{i}\epsilon)}$ as $\epsilon \to 0$, $\epsilon > 0$, exists and

$$\pm \lim_{\substack{\epsilon\to 0\\ \epsilon>0}} (-1)^n \mathrm{Im} \sqrt[c]{\chi_1(\lambda \mp \mathrm{i}\epsilon)} \geq 0.$$

Extending $\sqrt[c]{\chi_1(\lambda)}$ to $G_{1,n}$ from below one has $(-1)^{n+1}\mathrm{i}\sqrt[c]{\chi_1(\lambda)} > 0$ for any $\lambda^-_{1,n-1} < \lambda < \lambda^+_{1,n}$.

(iii) For any $\lambda \in \mathbb{R}$ with $\lambda < -\lambda^+_0(q,p)$ or $\lambda > \lambda^+_{-1}(q,p)$

$$\sqrt[c]{\chi_1(-\frac{1}{16\lambda}, -q, p)} > 0. \tag{6.54}$$

(iv) $\sqrt[c]{\chi_1(0,q,p)} = \sqrt[+]{\lambda^+_0\lambda^-_0} \prod_{n\geq 1} \frac{\lambda^+_n\lambda^-_n}{\pi^2_n}$.

Proof. Since by Lemma 6.18(ii), $(-1)^n f_n(\lambda,q,p) > 0$ for any $\lambda^+_{1,n-1} < \lambda < \lambda^-_{1,n+1}$ item (i) and (ii) follow from (6.48). Concerning item (iii) note that by item(i), $\sqrt[c]{\chi_1(-\frac{1}{16\lambda}, -q,p)} > 0$ for $\lambda^+_{1,-1}(-q,p) < -\frac{1}{16\lambda} < \lambda^-_{1,0}(-q,p)$. Since $\lambda^+_{1,-1}(-q,p) = -\lambda^-_1(-q,p) = -(16\lambda^+_{-1}(q,p))^{-1}$ and $\lambda^-_{1,0}(-q,p) = \lambda^-_0(-q,p) = (16\lambda^+_0(q,p))^{-1}$ one has

$$\lambda^+_{1,-1}(-q,p) < -\frac{1}{16\lambda} < \lambda^-_{1,0}(-q,p) \quad \text{iff} \quad [\lambda > \lambda^+_{-1}(q,p) \text{ or } -\lambda < \lambda^+_0(q,p)].$$

Item(iv) follows from (i) and (6.36). □

For any $v = (q,p) \in \hat{W}$ and $-\frac{1}{16\lambda} \in \mathbb{C}^* \setminus G_{1,n}(-q,p)$, $n \in \mathbb{Z}$,

$$w_{1,n}(-\frac{1}{16\lambda}, -q, p) = \sqrt[s]{(\lambda^+_{1,n}(-q,p) + \frac{1}{16\lambda})(\lambda^-_{1,n}(q,p) + \frac{1}{16\lambda})}.$$

Since $\lambda^\pm_n(-q,p) = \frac{1}{16\lambda^\mp_{-n}(q,p)}$ for any $n \in \mathbb{Z}$ it follows from the definition of $\lambda^\pm_{2,n}$ that

$$w_{1,n}(-\frac{1}{16\lambda}, -q, p) = \sqrt[s]{(\lambda^+_{2,n}(-q,p) + \frac{1}{16\lambda})(\lambda^-_{2.n}(q,p) + \frac{1}{16\lambda})}.$$

We define

$$w_{2,n} \equiv w_{2,n}(\lambda, v) := \sqrt[s]{(\lambda^+_{2,n}(v) + \frac{1}{16\lambda})(\lambda^-_{2,n}(v) + \frac{1}{16\lambda})}$$

and the canonical root $\sqrt[c]{\chi_2}$ of $\chi_2(\lambda) \equiv \chi_{p,2}(\lambda)$

$$\sqrt[c]{\chi_2(\lambda)} \equiv \sqrt[c]{\chi_2(\lambda, v)} := \prod_{n\in\mathbb{Z}} \frac{w_{2,n}(\lambda, v)}{\pi_n}.$$

For any $n \in \mathbb{Z}$, we have

$$w_{2,n}(\lambda, q, p) = w_{1,n}(-\frac{1}{16\lambda}, -q, p), \quad \text{and} \quad \sqrt[c]{\chi_2(\lambda, q, p)} = \sqrt[c]{\chi_1(-\frac{1}{16\lambda}, -q, p)}. \tag{6.55}$$

The canonical root $\sqrt[c]{\chi_p(\lambda)} \equiv \sqrt[c]{\chi_p(\lambda, v)}$ is then defined on $\hat{W}$ by

$$\sqrt[c]{\chi_p(\lambda)} := \mathrm{i}\frac{1}{\sqrt[c]{\chi_1(0)}} \sqrt[c]{\chi_1(\lambda)}\sqrt[c]{\chi_2(\lambda)}. \tag{6.56}$$

where $\lambda \in \mathbb{C}^* \setminus \bigcup_{m\in\mathbb{Z}} \big(G_{1,m}(v) \cup G_{2,m}(v)\big) = \mathbb{C}^* \setminus \bigcup_{m\in\mathbb{Z}}(G_m(v) \cup -G_m(v))$. Note that

$$G_{2,m}(q,p) = \{ -\frac{1}{16\lambda} \;:\; \mu \in G_{1,m}(-q,p) \} \tag{6.57}$$

and by (6.52)

$$\sqrt[c]{\chi_1(0)} = \sqrt[c]{\chi_2(\infty)}. \tag{6.58}$$

Lemma 6.21 *(i) For any $v_0 \in \hat{W}$, the canonical root $\sqrt[c]{\chi_p(\lambda)}$ is an analytic function in $\lambda \in \mathbb{C}^* \setminus \bigcup_{m\in\mathbb{Z}}(G_m \cup -G_m)$ and analytic in (λ, v) on $(\mathbb{C}^* \setminus \bigcup_{m\in\mathbb{Z}}(U_m \cup -U_m)) \times V_{v_0}$. In case $\gamma_m = 0$ for some $m \in \mathbb{Z}$, $\sqrt[c]{\chi_p(\lambda)}$ extends analytically to $\lambda = \tau_m$ and $\lambda = -\tau_m$.*

(ii) The canonical root at the zero potential is

$$\sqrt[c]{\chi_p(\lambda, 0)} = -\mathrm{i}\sin(\omega(\lambda)). \tag{6.59}$$

(iii) For any $v = (q,p) \in \hat{W}$ and $\lambda \in \mathbb{C}^* \setminus \big(\bigcup_{m\in\mathbb{Z}} G_m \cup -G_m\big)$ the following identites hold:

$$\sqrt[c]{\chi_p(-\lambda, v)} = -\sqrt[c]{\chi_p(\lambda, v)} \quad \text{and} \quad \sqrt[c]{\chi_p(-(16\,\lambda)^{-1}, -q, p)} = \sqrt[c]{\chi_p(\lambda, q, p)}. \tag{6.60}$$

Proof. Item (i) follows from Lemma 6.14 and Lemma 6.19. To prove (ii) note that for by Lemma 2.16, $\chi_p(\lambda, 0) = -\sin^2(\omega(\lambda))$. Hence $\sqrt[c]{\chi_p(\lambda, 0)} = \pm\mathrm{i}\sin(\omega(\lambda))$ and it remains to determine the sign. To this end note that for $v = 0$, $\lambda_n^+ = \lambda_n^-$ and hence $w_{1,n}(\lambda) = \tau_{1,n} - \lambda$ for any $n \in \mathbb{Z}$. One then concludes from (6.37) and the definition (6.51) of the c-root of $\chi_p(\lambda)$ that $\sqrt[c]{\chi_1(\lambda)} = \prod_{n\in\mathbb{Z}} \frac{\tau_{1,n}-\lambda}{\pi_n}$, $\lambda \in \mathbb{C}$. It implies that $\sqrt[c]{\chi_1(0)} = \prod_{n\in\mathbb{Z}} \frac{\tau_{1,n}}{\pi_n}$. As $\sqrt[c]{\chi_1(-\frac{1}{16\lambda})} = \sqrt[c]{\chi_2(\lambda)}$ one has $\sqrt[c]{\chi_2(\infty)} = \sqrt[c]{\chi_1(0)}$. Since $\tau_{1,n} = n\pi + \ell_n^2$ and $\omega(\lambda) = \lambda - \frac{1}{16\lambda}$ it then follows from the definition (6.56) of $\sqrt[c]{\chi_p(\lambda)}$ and Lemma B.5 that uniformly for $\lambda \in \partial B_N$

$$\sqrt[c]{\chi_p(\lambda, 0)} = -\mathrm{i}(1 + o(1))\sin(\omega(\lambda)) \quad \text{as } N \to \infty$$

and uniformly for $\lambda \in \partial B_{-N}$

$$\sqrt[c]{\chi_p(\lambda, 0)} = -\mathrm{i}(1 + o(1))\sin(\omega(\lambda)) \quad \text{as } N \to \infty.$$

Since $\sqrt[c]{\chi_p(\lambda, 0)}$ and $\sin(\omega(\lambda))$ have the same roots it then follows from Lemma C.1 that

$$\sqrt[c]{\chi_p(\lambda, 0)} = -\mathrm{i}\sin(\omega(\lambda)).$$

(iii) The identities of Δ stated in Lemma 2.14 imply the claimed symmetries hold up to a sign. Furthermore, since $\omega(-\lambda) = -\omega(\lambda)$ and $\omega(-\frac{1}{16\,\lambda}) = \omega(\lambda)$, they hold for $v = 0$. Hence by continuity they hold on $\hat{W}$. □

On H_r^1, the sign table for $\sqrt[c]{\chi_p(\lambda)}$ can be computed by using Lemma 6.20.

Lemma 6.22 *On H_r^1, for any $n \in \mathbb{Z}$, the following holds:*

(i) For any $\lambda_{1,n-1}^+ < \lambda < \lambda_{1,n}^-$

$$(-1)^n \mathrm{Im}\sqrt[c]{\chi_p(\lambda)} > 0. \tag{6.61}$$

Similarly, for any $\lambda_{2,n-1}^+ < -\frac{1}{16\lambda} < \lambda_{2,n}^-$

$$(-1)^n \mathrm{Im}\sqrt[c]{\chi_p(\lambda)} > 0. \tag{6.62}$$

(ii) For any $\lambda_{1,n}^- \le \lambda \le \lambda_{1,n}^+$, the limits of $\sqrt[c]{\chi_p(\lambda + \mathrm{i}\epsilon)}$ and $\sqrt[c]{\chi_p(\lambda - \mathrm{i}\epsilon)}$ as $\epsilon \to 0$, $\epsilon > 0$, exist and

$$\pm \lim_{\substack{\epsilon\to 0\\ \epsilon>0}} (-1)^n \mathrm{Re}\sqrt[c]{\chi_p(\lambda \pm \mathrm{i}\epsilon)} \ge 0.$$

Figure 3: Illustration of the sign of $\sqrt[c]{\Delta^2 - 1}$

Extending $\sqrt[c]{\chi_p(\lambda)}$ *continuously to* $G_{1,n}$ *from below one has*

$$(-1)^{n+1}\sqrt[c]{\chi_p(\lambda)} > 0, \quad \forall\ \lambda_{1,n}^- < \lambda < \lambda_{1,n}^+. \tag{6.63}$$

Similarly, for any $\lambda_{2,n}^- < -\frac{1}{16\lambda} < \lambda_{2,n}^+$ *the limit* $\lim_{\substack{\epsilon\to 0\\ \epsilon>0}} \sqrt[c]{\chi_p(\lambda - \mathrm{i}\epsilon)}$ *exists and if one extends* $\sqrt[c]{\chi_p(\lambda)}$ *to* $G_{2,n}$ *from below, then*

$$(-1)^{n+1}\sqrt[c]{\chi_p(\lambda)} > 0, \quad \forall\ \lambda_{2,n}^- < -\frac{1}{16\lambda} < \lambda_{2,n}^+. \tag{6.64}$$

Proof. (i) For any $\lambda_{1,n-1}^+ < \lambda < \lambda_{1,n}^-$ one has according to Lemma 6.20,

$$(-1)^n \sqrt[c]{\chi_1(\lambda)} > 0, \quad \sqrt[c]{\chi_1(0)} > 0, \quad \text{and } \sqrt[c]{\chi_1(-\frac{1}{16\lambda}, -q, p)} > 0.$$

By the definition of $\sqrt[c]{\chi_p(\lambda)}$ it then follows that $(-1)^n \mathrm{Im}\sqrt[c]{\chi_p(\lambda)} > 0$. Next consider $\lambda_{2,n-1}^+ < -\frac{1}{16\lambda} < \lambda_{2,n}^-$. Since $\lambda_{2,n-1}^+(q,p) = \lambda_{1,n-1}^+(-q,p)$ and $\lambda_{2,n}^-(q,p) = \lambda_{1,n}^-(-q,p)$ one has $\lambda_{1,n-1}^+(-q,p) < -\frac{1}{16\lambda} < \lambda_{1,n}^-(-q,p)$ and thus by (6.61), $(-1)^n \mathrm{Im}\sqrt[c]{\chi_p(-\frac{1}{16\lambda}, -q, p)} > 0$. Since by Lemma 6.21(iii), $\sqrt[c]{\chi_p(-\frac{1}{16\lambda}, -q, p)} = \sqrt[c]{\chi_p(\lambda, q, p)}$, the claimed inequality follow.
(ii) The claimed result follow from Lemma 6.20. In particular, for any $\lambda_{1,n}^- < \lambda < \lambda_{1,n}^+$, one has by Lemma 6.20, $\sqrt[c]{\chi_1(0)} > 0$, $\sqrt[c]{\chi_1(-\frac{1}{16\lambda}, -q, p)} > 0$, and $(-1)^{n+1}\mathrm{i}\sqrt[c]{\chi_1(\lambda)} > 0$. By the definition of $\sqrt[c]{\chi_p(\lambda)}$ it then follows that when $\sqrt[c]{\chi_p(\lambda)}$ is extended continuously to $G_{1,n}$ from below then $(-1)^{n+1}\sqrt[c]{\chi_p(\lambda)} > 0$, for any $\lambda_{1,n}^- < \lambda < \lambda_{1,n}^+$. Next consider the case where $\lambda_{2,n}^- < -\frac{1}{16\lambda} < \lambda_{2,n}^+$. (It means that in case $n \geq 0$, $-\lambda_{-n}^+ < \lambda < -\lambda_{-n}^-$ whereas in case $n \leq -1$, $\lambda_{-n}^- < \lambda < \lambda_{-n}^+$.) Use again that $\lambda_{2,n}^\pm(q,p) = \lambda_{1,n}^\pm(-q,p)$ to conclude by the above that $(-1)^{n+1}\sqrt[c]{\chi_p(-\frac{1}{16\lambda}, -q, p)} > 0$. Since $\sqrt[c]{\chi_p(-\frac{1}{16\lambda}, -q, p)} = \sqrt[c]{\chi_p(\lambda, q, p)}$ it then follows that $(-1)^n \sqrt[c]{\chi_p(\lambda, q, p)} > 0$ for any $\lambda_{2,n}^- < -\frac{1}{16\lambda} < \lambda_{2,n}^+$. □

We finish this section with various asymptotic estimates.

Lemma 6.23 *(i) Uniformly for* $\lambda \in U_{1,m}$ *and locally uniformly for* $v \in \hat{W}$,

$$\frac{w_{1,m}(\lambda)}{\sqrt[c]{\chi_1(\lambda)}} \frac{-\sin(\omega(\lambda))}{\pi_m - \omega(\lambda)}, \ \frac{w_{1,m}(\lambda)}{\sqrt[c]{\chi_p(\lambda)}} \frac{-\mathrm{i}\sin(\omega(\lambda))}{\pi_m - \omega(\lambda)} = 1 + \ell_m^2 \quad \text{as } |m| \to \infty \tag{6.65}$$

(ii) Uniformly for $\lambda \in \partial B_N$ *and locally uniformly for* $v \in \hat{W}$

$$\frac{-\sin(\omega(\lambda))}{\sqrt[c]{\chi_1(\lambda)}}, \ \frac{-\mathrm{i}\sin(\omega(\lambda))}{\sqrt[c]{\chi_p(\lambda)}} = 1 + o(1) \quad \text{as } N \to \infty \tag{6.66}$$

(iii) Uniformly for $\lambda \in U_{2,m}$ *and locally uniformly for* $v \in \hat{W}$

$$\frac{w_{2,m}(\lambda)}{\sqrt[c]{\chi_2(\lambda)}} \frac{-\sin(\omega(\lambda))}{\pi_m - \omega(\lambda)}, \ \frac{w_{2,m}(\lambda)}{\sqrt[c]{\chi_p(\lambda)}} \frac{-\mathrm{i}\sin(\omega(\lambda))}{\pi_m - \omega(\lambda)} = 1 + \ell_m^2 \quad \text{as } |m| \to \infty \tag{6.67}$$

(iv) Uniformly for $\lambda \in \partial B_{-N}$ and locally uniformly for $v \in \hat{W}$

$$\frac{-\sin(\omega(\lambda))}{\sqrt[c]{\chi_2(\lambda)}}, \ \frac{-\mathrm{i}\sin(\omega(\lambda))}{\sqrt[c]{\chi_p(\lambda)}} = 1 + o(1) \quad \text{as } N \to \infty \tag{6.68}$$

Proof. (i) Since $\sqrt[c]{\chi_p(\lambda)} = \mathrm{i}\sqrt[c]{\chi_1(\lambda)}\sqrt[c]{\chi_2(\lambda)}/\sqrt[c]{\chi_2(\infty)}$, it remains to verify the claimed asymptotics for $\frac{w_{1,m}(\lambda)}{\sqrt[c]{\chi_p(\lambda)}}\frac{-\mathrm{i}\sin(\omega(\lambda))}{\pi_m-\omega(\lambda)}$. For $v \in \hat{W}$ and $\lambda \in U_{1,m}$, it follows from the definitions (6.51) and (6.56) of the canonical and standard roots that

$$\frac{\sqrt[c]{\chi_p(\lambda)}}{w_{1,m}(\lambda)} = \mathrm{i}\frac{\sqrt[c]{\chi_1(-(16\lambda)^{-1}, -q, p)}}{\sqrt[c]{\chi_1(0,q,p)}} f_{1,m}(\lambda, q, p)$$

where by (6.50), $f_{1,m}(\lambda) \equiv f_{1,m}(\lambda, v)$ is given by

$$f_{1,m}(\lambda) = \frac{1}{\pi_m}\prod_{k \neq m}\frac{w_{1,k}(\lambda)}{\pi_k}.$$

Let us first consider the quotient $\sqrt[c]{\chi_1(-(16\lambda)^{-1}, -q, p)}/\sqrt[c]{\chi_1(0,q,p)}$. Since $\sqrt[c]{\chi_1(\lambda, -q, p)}$ does not vanish at $\lambda = 0$ and is differentiable (cf definition (6.4) of $\hat{W}$) and since $\sqrt[c]{\chi_1(0,q,p)} = \sqrt[c]{\chi_1(0,-q,p)}$ (cf (6.51)) one sees by expanding $\sqrt[c]{\chi_1(\mu, -q, p)}$ at $\mu = 0$ that for $\lambda \in U_{1,m}$ and hence $\lambda^{-1} = \ell^2_m$, (cf (I-2) of Section 6.2)

$$\frac{\sqrt[c]{\chi_1(-(16\lambda)^{-1}, -q, p)}}{\sqrt[c]{\chi_1(0,q,p)}} = 1 + \ell^2_m \quad \text{as} \quad |m| \to \infty$$

and in turn

$$\frac{\sqrt[c]{\chi_p(\lambda)}}{w_{1,m}(\lambda)} = \mathrm{i}f_{1,m}(\lambda)(1 + \ell^2_m) \quad \text{as} \quad |m| \to \infty.$$

In particular, for $v = 0$ one obtains, taking into account Lemma 6.21(ii)

$$\frac{-\mathrm{i}\sin(\omega(\lambda))}{\pi_m - \omega(\lambda)} = \frac{\sqrt[c]{\chi_p(\lambda, 0)}}{\pi_m - \omega(\lambda)} = \mathrm{i}f_{1,m}(\lambda, 0)(1 + \ell^2_m)$$

implying that

$$\frac{w_{1,m}(\lambda)}{\sqrt[c]{\chi_p(\lambda)}}\frac{-\mathrm{i}\sin(\omega(\lambda))}{\pi_m - \omega(\lambda)} = \frac{f_{1,m}(\lambda, 0)}{f_{1,m}(\lambda)}(1 + \ell^2_m) \quad \text{as} \quad |m| \to \infty.$$

It remains to analyze the asymptotics of $f_{1,m}(\lambda)$, $\lambda \in U_{1,m}$, as $|m| \to \infty$. By property (I-3) of Section 6.2 there exists $M \geq 1$ so that for any $|m| \geq M$ and $\lambda \in U_{1,m}$,

$$w_{1,k}(\lambda) = (\tau_{1,k} - \lambda)\sqrt[+]{1 - \gamma^2_{1,k}/4(\tau_{1,k} - \lambda)^2}, \quad k \neq m.$$

Hence

$$\frac{f_{1,m}(\lambda, 0)}{f_{1,m}(\lambda)} = \prod_{k \neq m}\frac{\pi_k - \lambda}{\tau_{1,k} - \lambda}\frac{1}{\sqrt[+]{1 - \gamma^2_{1,k}/4(\tau_{1,k} - \lambda)^2}}.$$

By Lemma B.3, it then follows that uniformly for $\lambda \in U_{1,m}$

$$\frac{f_{1,m}(\lambda, 0)}{f_{1,m}(\lambda)} = 1 + \ell^2_m \quad \text{as} \quad |m| \to \infty$$

and hence (6.65) is proved. Going trough the arguments of the proof one sees that (6.65) holds locally uniformly on $\hat{W}$.

(ii) Arguing in a similar way as in the proof of item(i) one shows that the claimed asymptotics follow from Lemma B.5. The claimed asymptotics of item (iii) and (iv) follow by reciprocity (cf (6.60)),

$$\sqrt[c]{\chi_p(-(16\lambda)^{-1}, -q, p)} = \sqrt[c]{\chi_p(\lambda, q, p)}.$$

$\square$

When combined with the asymptotics of infinite products of Lemma B.4 and Lemma B.5, Lemma 6.23 leads to the following application. Recall that

$$\ell^* := \{ \ \sigma = (\sigma_n)_n \subset \mathbb{C}^* \ : \ (\sigma_m - m\pi)_m \in \ell^2 \ \}.$$

***Corollary* 6.24** *(i) Locally uniformly for $v \in \hat{W}$ and $\sigma \in \ell^*$*

$$\sup_{\lambda \in U_{1,n}} \Big| \prod_{m \neq n} \frac{\sigma_m - \lambda}{w_{1,m}(\lambda)} - 1 \Big| = \ell^2_n.$$

(ii) Locally uniformly for $v \in \hat{W}$ and $\sigma \in \ell^*$

$$\sup_{\lambda \in \partial B_N} \Big| \prod_{m \in \mathbb{Z}} \frac{\sigma_m - \lambda}{w_{1,m}(\lambda)} - 1 \Big| = o(1) \quad \text{as } N \to \infty.$$

7 Invariant tori

The main purpose of this chapter is to show that the 1-parameter family $\Delta_\lambda(v)$, $\lambda \in \mathbb{C}^*$, given by the discriminant, $\Delta_\lambda(v) = \Delta(\lambda, v)$, is a family of integrals of the sinh-Gordon equation (Corollary 7.6) which Poisson commute (Theorem 7.4). In addition we analyze for any $v_0 \in H^1_r$, the isospectral set of v_0,

$$Iso(v_0) := \{ \ v \in H^1_r \ : \ \Delta_\lambda(v) = \Delta_\lambda(v_0) \ \ \forall \ \lambda \in \mathbb{C}^* \ \}.$$

7.1 Poisson brackets

Recall that by (1.6), the Poisson bracket of two given C^1- functionals $F, G : H^1_c \to \mathbb{C}$ is given by

$$\{F, G\} = -\int_0^1 \partial_v F J P^{-1} \partial_v G \, \mathrm{d}x \tag{7.1}$$

where $\partial_v F = (\partial_q F, \partial_p F)$ denotes the L^2-gradient of F. Here $\partial_v F$ and $\partial_v G$ are assumed to be sufficiently regular so that (7.1) is well defined. In the sequel, we consider functionals $F : H^1_c \to \mathbb{C}$ satisfying

$$\partial_q F \in L^2_{\mathbb{C}}, \quad \partial_p F \in H^{-1}_{\mathbb{C}} \tag{7.2}$$

If F, G are functionals both satisfying (7.2) one has

$$\{F, G\} = -\int_0^1 \big(\partial_p G P^{-1} \partial_q F - \partial_p F P^{-1} \partial_q G\big) \, \mathrm{d}x \tag{7.3}$$

where $\int_0^1 \partial_p G P^{-1} \partial_q F \, \mathrm{d}x$ stands for the $H^{-1}_{\mathbb{C}}$, $H^1_{\mathbb{C}}$ parring, $\int_0^1 \partial_p G P^{-1} \partial_q F \, \mathrm{d}x = \langle \partial_p G, P^{-1} \partial_q F \rangle_r$ which is well defined since $\partial_p G \in H^{-1}_{\mathbb{C}}$ and $\partial_q F \in H^0_{\mathbb{C}}$, implying that $P^{-1} \partial_q F \in H^1_{\mathbb{C}}$.

More generally, consider $\tilde{a}lsof \in H^{-1}_{\mathbb{C}}$ of the form

$$\tilde{f} = (f_1, f_2 P(\cdot)), \quad f = (f_1, f_2) \in L^2_c. \tag{7.4}$$

For $\tilde{f}, \tilde{g} \in H^{-1}_c$ both satisfying (7.4) we introduce the skew symmetric bilinear form

$$[\tilde{f}, \tilde{g}]_1 = -\int_0^1 (f_1 g_2 - f_2 g_1) \, \mathrm{d}x = -\int_0^1 f \cdot J g \, \mathrm{d}x. \tag{7.5}$$

Furthermore recall that we introduced $[\![f]\!]_{q,\lambda}$ for any given $q \in H^1_{\mathbb{C}}$, $\lambda \in \mathbb{C}^*$, and $f = (f_1, f_2) \in H^1_{\mathbb{C}}([0,1]) \times H^1_{\mathbb{C}}([0,1])$. Actually, it suffices to assume, that $f_j \in L^4_{\mathbb{C}}([0,1])$, $j = 1, 2$, to assure that $[\![f]\!]_{q,\lambda} = ([\![f]\!]_{q,\lambda,1}, [\![f]\!]_{q,\lambda,2} P(\cdot))$ is an element in H^{-1}_c, satisfying (7.4), where

$$[\![f]\!]_{q,\lambda,1} = \frac{\lambda}{2}(f_2^2 - f_1^2) + \frac{1}{32\lambda}(f_2^2 e^q - f_1^2 e^{-q}), \quad [\![f]\!]_{q,\lambda,2} = -\frac{1}{2} f_1 f_2.$$

It is also useful to introduce $[\![f]\!]^{\sim}_{q,\lambda} := ([\![f]\!]_{q,\lambda,1}, [\![f]\!]_{q,\lambda,2})$. Recall that we denoted by M_1, M_2 the columns of the fundamental matrix $M = M(x, \lambda, v)$.

Lemma 7.1 *Let $\lambda, \mu \in \mathbb{C}^*$, $v \in H^1_c$ and $f = a_1 M_1\big|_\lambda + a_2 M_2\big|_\lambda$, $g = b_1 M_1\big|_\mu + b_2 M_2\big|_\mu$, where $a = (a_1, a_2)$, $b = (b_1, b_2) \in \mathbb{C}^2$. Then the following holds:*

(i) If $\lambda \neq \pm\mu$,then

$$16\big[\, [\![f]\!]^{\sim}_{q,\lambda}\,,[\![g]\!]^{\sim}_{q,\mu}\,\big]_1 = -\frac{\lambda+\mu}{\lambda-\mu}(f\cdot Jg)^2\Big|_0^1 + \frac{\lambda-\mu}{\lambda+\mu}(f\cdot Zg)^2\Big|_0^1.$$

(ii) If $\lambda = \mu$, then

$$16\big[\, [\![f]\!]^{\sim}_{q,\lambda}\,,[\![g]\!]^{\sim}_{q,\lambda}\,\big]_1 = 4\lambda(a_1b_2 - a_2b_1)(f\cdot J\partial_\lambda g)\big|_{\lambda,x=1}.$$

Proof. (i) The result follows from the following computation

$$\begin{aligned}
4\big[\, [\![f]\!]^{\sim}_{q,\lambda}\,,[\![g]\!]^{\sim}_{q,\mu}\,\big]_1 = &\int_0^1 g_1g_2\left(\lambda(f_2^2 - f_1^2) + \frac{1}{16\lambda}(f_2^2e^q - f_1^2e^{-q})\right)\\
&- f_1f_2\left(\mu(g_2^2 - g_1^2) + \frac{1}{16\mu}(g_2^2e^q - g_1^2e^{-q})\right)\mathrm{d}x\\
=&\int_0^1 (g_1g_2\frac{1}{16\lambda}f_2^2 - f_1f_2\frac{1}{16\mu}g_2^2)e^q + (f_1f_2\frac{1}{16\mu}g_1^2 - g_1g_2\frac{1}{16\lambda}f_1^2)e^{-q}\\
&+ \lambda\mu(\frac{g_1f_2}{\mu}g_2f_2 - \frac{f_1g_2}{\lambda}g_2f_2) + \lambda\mu(\frac{g_1f_2}{\lambda}f_1g_1 - \frac{f_1g_2}{\mu}f_1g_1)\,\mathrm{d}x
\end{aligned}$$

It is convenient to introduce

$$L_1 := \frac{f_1g_2}{\lambda} - \frac{f_2g_1}{\mu},\quad L_2 := \frac{f_1g_2}{\mu} - \frac{f_2g_1}{\lambda},\quad L_3 =: f_1g_1,\quad L_4 := f_2g_2.$$

Then

$$4\big[\, [\![f]\!]^{\sim}_{q,\lambda}\,,[\![g]\!]^{\sim}_{q,\mu}\,\big]_1 = -\int_0^1 L_2L_4\frac{e^q}{16} + L_1L_3\frac{e^{-q}}{16} + \lambda\mu(L_1L_4 + L_2L_3)\,\mathrm{d}x. \tag{7.6}$$

We claim that the integrand of the latter integral is a full derivative. To see it we compute the derivatives of L_1 and L_2. Since $L_1 = f\cdot\begin{pmatrix} & 1/\lambda\\ -1/\mu & \end{pmatrix} g$ we obtain by the Leibniz rule

$$\begin{aligned}
\partial_x L_1 =& J(\lambda - A - B^2/\lambda)f\cdot\begin{pmatrix} & \frac{1}{\lambda}\\ -\frac{1}{\mu} & \end{pmatrix}g + f\cdot\begin{pmatrix} & \frac{1}{\lambda}\\ -\frac{1}{\mu} & \end{pmatrix}J(\mu - A - B^2/\mu)g\\
=& f\cdot\Big((\lambda - A - B^2/\lambda)J^t\begin{pmatrix} & \frac{1}{\lambda}\\ -\frac{1}{\mu} & \end{pmatrix} + \begin{pmatrix} & \frac{1}{\lambda}\\ -\frac{1}{\mu} & \end{pmatrix}J(\mu - A - B^2/\mu)\Big)g
\end{aligned}$$

yielding

$$\partial_x L_1 = f\cdot\begin{pmatrix}\frac{\lambda^2-\mu^2}{\lambda\mu} & \\ & \frac{e^q}{16}\frac{\lambda^2-\mu^2}{\lambda^2\mu^2}\end{pmatrix}g = \frac{\lambda^2-\mu^2}{\lambda^2\mu^2}\Big(\lambda\mu L_3 + \frac{e^q}{16}L_4\Big). \tag{7.7}$$

Similarly one computes

$$\partial_x L_2 = \frac{\lambda^2-\mu^2}{\lambda^2\mu^2}\left(\lambda\mu L_4 + \frac{e^{-q}}{16}L_3\right). \tag{7.8}$$

Substituting (7.7)-(7.8) into (7.6) one gets

$$\begin{aligned}
4\frac{\lambda^2-\mu^2}{\lambda^2\mu^2}\big[\, [\![f]\!]^{\sim}_{q,\lambda}\,,[\![g]\!]^{\sim}_{q,\mu}\,\big]_1 =& -\int_0^1 L_2\frac{\lambda^2-\mu^2}{\lambda^2\mu^2}(\lambda\mu L_3 + \frac{e^q}{16}L_4) + L_1\frac{\lambda^2-\mu^2}{\lambda^2\mu^2}(\lambda\mu L_4 + \frac{e^{-q}}{16}L_3)\\
=& -\int_0^1 (L_2\partial_x L_1 + L_1\partial_x L_2)\,\mathrm{d}x = -(L_1L_2)\Big|_0^1.
\end{aligned}$$

Since

$$\begin{aligned}
4L_1L_2 =& (L_1+L_2)^2 - (L_1-L_2)^2 = (\frac{1}{\lambda}+\frac{1}{\mu})^2(f\cdot Jg)^2 - (\frac{1}{\lambda}-\frac{1}{\mu})^2(f\cdot Zg)^2\\
=& \frac{(\lambda+\mu)^2}{\lambda^2\mu^2}(f\cdot Jg)^2 - \frac{(\lambda-\mu)^2}{\lambda^2\mu^2}(f\cdot Zg)^2
\end{aligned}$$

we then obtain the claimed identity

$$16\big[\,[\![f]\!]^{\sim}_{q,\lambda}\,,[\![g]\!]^{\sim}_{q,\mu}\,\big]_1 = -\frac{\lambda+\mu}{\lambda-\mu}(f\cdot Jg)^2\Big|_0^1 + \frac{\lambda-\mu}{\lambda+\mu}(f\cdot Zg)^2\Big|_0^1.$$

(ii) Note that for $\lambda=\mu$ one has by the Wronskian identity for all $x\in\mathbb{R}$

$$(a_1M_1+a_2M_2)\cdot J(b_1M_1+b_2M_2) = a_1b_2-a_2b_1. \tag{7.9}$$

Hence $(f\cdot Jg)^2\Big|_0^1 = 0$. This implies that the analytic function

$$\Psi:\mathbb{C}\setminus\{\lambda\}\to\mathbb{C},\ \mu\mapsto -\frac{\lambda+\mu}{\lambda-\mu}(f\big|_\lambda\cdot Jg\big|_\mu)^2\Big|_0^1 + \frac{\lambda-\mu}{\lambda+\mu}(f\big|_\lambda\cdot Zg\big|_\mu)^2\Big|_0^1.$$

has a removable singularity in $\mu=\lambda$ and hence can be extended analytically to λ by

$$\Psi(\lambda) = 2\lambda\partial_\mu(f\big|_\lambda\cdot Jg\big|_\mu)^2\Big|_0^1\Big|_{\mu=\lambda}.$$

Since $\partial_\mu M(0,\mu,v)\equiv 0$ it follows that

$$\partial_\mu(f\big|_\lambda\cdot Jg\big|_\mu)^2\Big|_0^1 = 2(f\big|_\lambda\cdot Jg\big|_\mu)(f\big|_\lambda\cdot J\partial_\mu g\big|_\mu)\big|_{x=1}$$

By (7.9) it then follows that

$$\Psi(\lambda) = 4\lambda(a_1b_2-a_2b_1)(f\cdot J\partial_\lambda g)\big|_{x=1}.$$

Since $16\big[\,[\![f]\!]^{\sim}_{q,\lambda}\,,[\![g]\!]^{\sim}_{q,\mu}\,\big]_1$ is analytic in μ the claimed identity follows. □

We now apply Lemma 7.1 to show that Δ_λ and Δ_μ Poisson commute on H^1_c for any $\lambda,\mu\in\mathbb{C}^*$, where $\Delta_\lambda(v)\equiv\Delta(\lambda,v)$ is the discriminant introduced in (2.33). First we want to write $\partial_v\Delta_\lambda$ in a form which allows to apply Lemma 7.1. By Lemma 5.2(i) on has

$$\partial_v\Delta_\lambda = (f_{\lambda,1}, f_{\lambda,2}P(\cdot)),\quad f_\lambda := (f_{\lambda,1}, f_{\lambda,2})$$

where with $\delta_\lambda=\delta(\lambda)$ given by (2.33),

$$\begin{aligned} f_{\lambda,1} :=& \partial_q\Delta_\lambda = \frac{\lambda}{4}\big(\grave{m}_2(m_3^2-m_1^2)+\grave{m}_3(m_2^2-m_4^2)+2\delta_\lambda(m_1m_2-m_3m_4)\big)\\ &+\frac{1}{64\lambda}\big(e^{-q}(\grave{m}_3m_2^2-\grave{m}_2m_1^2+2\delta_\lambda m_1m_2)+e^{q}(\grave{m}_2m_3^2-\grave{m}_3m_4^2-2\delta_\lambda m_3m_4)\big)\end{aligned} \tag{7.10}$$

and

$$f_{\lambda,2} := \frac{1}{4}\big(-\grave{m}_2m_1m_3+\grave{m}_3m_2m_4+\delta_\lambda(m_1m_4+m_2m_3)\big). \tag{7.11}$$

For later use we record the following

Lemma 7.2 *For any $\lambda\in\mathbb{C}^*$ and $v\in H^1_c$, $f_{\lambda,1},f_{\lambda,2}\in H^1_{\mathbb{C}}$. Hence $\partial_\lambda\Delta_\lambda=(f_{\lambda,1},Pf_{\lambda,2})\in L^2_c$.*

Proof. By Corollary 2.3, $f_{\lambda,1}(x), f_{\lambda,2}(x)$ are continuous in x and $\partial_x f_{\lambda,1},\partial_x f_{\lambda,2}\in L^2_{\mathbb{C}}$. Furthermore, one verifies in straightforward way that $f_{\lambda,1}$, $f_{\lambda,2}$ are 1-periodic, implying that $f_{\lambda,1},f_{\lambda,2}\in H^1_{\mathbb{C}}$. Since P is selfadjoint with respect to $\langle\cdot,\cdot\rangle_r$ one has for $g=(g_1,g_2)\in H^1_c$

$$\mathrm{d}_v\,\Delta_\lambda[g] = \langle\partial_v\Delta_\lambda,\ g\rangle_r = \int_0^1(f_{\lambda,1}g_1+P(f_{\lambda,2})g_2)\,\mathrm{d}x.$$

This means that $\partial_v\Delta_\lambda=(f_{\lambda,1},\ P(f_{\lambda,2}))$ and $\partial_v\Delta_\lambda\in L^2_c$. □

It follows from (7.3) and (7.5) that $\{\Delta_\lambda,\Delta_\mu\}$ is well defined on H^1_c and given by

$$\{\Delta_\lambda,\Delta_\mu\} = -\int_0^1 f_\lambda\cdot Jf_\mu\,\mathrm{d}x. \tag{7.12}$$

Next we need to represent f_λ as a sum of functions of the form $[\![g]\!]^{\sim}_{q,\lambda}$, where g is a linear combination of the columns $M_1(x,\lambda)$, $M_2(x,\lambda)$ of $M(x,\lambda)$. The following lemma can be proved by a straightforward computation. We write $\delta_\lambda(v)=\delta(\lambda,v)$.

***Lemma* 7.3** *For any $\lambda \in \mathbb{C}^*$, $v \in H_c^1$,*

$$2\grave{m}_2(\lambda) f_\lambda = [\![\grave{m}_2(\lambda) M_1(x,\lambda) - \delta_\lambda M_2(x,\lambda)]\!]^\sim_{q,\lambda} - (\Delta_\lambda^2 - 1)\, [\![M_2(x,\lambda)]\!]^\sim_{q,\lambda} .$$

***Theorem* 7.4** *For any $\lambda, \mu \in \mathbb{C}^*$, $\{\Delta_\lambda, \Delta_\mu\} = 0$ on H_c^1.*

Proof. Clearly $\{\Delta_\lambda, \Delta_\lambda\} = 0$ and since $\Delta_{-\lambda} = \Delta_\lambda$ by Lemma 2.14, one also has $\{\Delta_\lambda, \Delta_{-\lambda}\} = 0$. For the remainder of the proof we always will assume that $\lambda \notin \{\pm\mu\}$. By (7.12) we have

$$\begin{aligned}
-4\grave{m}_2(\lambda)\grave{m}_2(\mu)\{\Delta_\lambda, \Delta_\mu\} = & \int_0^1 2\grave{m}_2(\lambda) f_\lambda \cdot J 2\grave{m}_2(\mu) f_\mu \,\mathrm{d}x \\
= & \big[\, [\![\grave{m}_2(\lambda) M_1(x,\lambda) - \delta_\lambda M_2(x,\lambda)]\!]^\sim_{q,\lambda} , [\![\grave{m}_2(\mu) M_1(x,\mu) - \delta_\mu M_2(x,\mu)]\!]^\sim_{q,\mu} \,\big]_1 \\
& - (\Delta_\lambda^2 - 1)\big[\, [\![M_2(x,\lambda)]\!]^\sim_{q,\lambda} , [\![\grave{m}_2(\mu) M_1(x,\mu) - \delta_\mu M_2(x,\mu)]\!]^\sim_{q,\mu} \,\big]_1 \\
& - (\Delta_\mu^2 - 1)\big[\, [\![\grave{m}_2(\lambda) M_1(x,\lambda) - \delta_\lambda M_2(x,\lambda)]\!]^\sim_{q,\lambda} , [\![M_2(x,\mu)]\!]^\sim_{q,\mu} \,\big]_1 \\
& + (\Delta_\lambda^2 - 1)(\Delta_\mu^2 - 1)\big[\, [\![M_2(x,\lambda)]\!]^\sim_{q,\lambda} , [\![M_2(x,\mu)]\!]^\sim_{q,\mu} \,\big]_1 .
\end{aligned}$$

By Lemma 7.1 it then follows that

$$64\grave{m}_2(\lambda)\grave{m}_2(\mu)\{\Delta_\lambda, \Delta_\mu\} = \frac{\lambda+\mu}{\lambda-\mu} F\Big|_0^1 - \frac{\lambda-\mu}{\lambda+\mu} G\Big|_0^1$$

where $F(x) = I_F(x) - II_F(x) - III_F(x) + IV_F(x)$ with

$$\begin{aligned}
I_F(x) = & \Big((\grave{m}_2(\lambda) M_1(x,\lambda) - \delta_\lambda M_2(x,\lambda)) J (\grave{m}_2(\mu) M_1(x,\mu) - \delta_\mu M_2(x,\mu)) \Big)^2 \\
II_F(x) = & (\Delta_\lambda^2 - 1)\Big(M_2(x,\lambda) \cdot J\big(\grave{m}_2(\mu) M_1(x,\mu) - \delta_\mu M_2(x,\mu)\big) \Big)^2 \\
III_F(x) = & (\Delta_\mu^2 - 1)\Big(\big(\grave{m}_2(\lambda) M_1(x,\lambda) - \delta_\lambda M_2(x,\lambda)\big) \cdot J M_2(x,\mu) \Big)^2 \\
IV_F(x) = & (\Delta_\lambda^2 - 1)(\Delta_\mu^2 - 1)\Big(M_2(x,\lambda) \cdot J M_2(x,\mu) \Big)^2
\end{aligned}$$

and $G(x)$, $I_G(x)$, $II_G(x)$, $III_G(x)$, and $IV_G(x)$ are defined in the same way as $F(x) = I_F(x) + \cdots + IV_F(x)$ except that the matrix J is replaced by Z throughout. We show that $F(1) = F(0)$ and $G(1) = G(0)$. Let us first prove that $F(1) = F(0)$. Using that $\delta_\lambda^2 - \Delta_\lambda^2 + 1 = -\grave{m}_2(\lambda)\grave{m}_3(\lambda)$ one sees by a straightforward computation that

$$F(0) = -\,\grave{m}_2(\lambda)^2 \grave{m}_2(\mu)\grave{m}_3(\mu) - \grave{m}_2(\mu)^2 \grave{m}_3(\lambda)\grave{m}_3(\lambda) - 2\grave{m}_2(\lambda)\grave{m}_2(\mu)\delta_\lambda\delta_\mu .$$

To compute $F(1)$ first note that

$$\grave{m}_2(\lambda)\grave{M}_1(\lambda) - \delta_\lambda \grave{M}_2(\lambda) = \Delta_\lambda \grave{M}_2(\lambda) + \begin{pmatrix} 0 \\ -1 \end{pmatrix} .$$

One then expands $F(1)$ in a polynomial in a of degree 2, where a is given by

$$a := \grave{M}_2(\lambda) J \grave{M}_2(\mu) = \grave{m}_2(\lambda)\grave{m}_4(\mu) - \grave{m}_4(\lambda)\grave{m}_2(\mu).$$

It is easy to verify that the coefficient of a^2 equals 1. Combining the a^2 with the term containing a then yields

$$\begin{aligned}
F(1) = & (\grave{m}_1(\lambda)\grave{m}_2(\mu) - \grave{m}_2(\lambda)\grave{m}_1(\mu))a - (\Delta_\mu^2 - 1)\grave{m}_2(\mu)^2 \\
& + (\Delta_\mu \grave{m}_2(\mu) - \Delta_\lambda \grave{m}_2(\lambda))^2 - (\Delta_\lambda^2 - 1)\grave{m}_2(\lambda)^2 .
\end{aligned}$$

Expanding this expression further one finally sees that $F(1) = F(0)$. Similarly one shows that $G(1) = G(0)$.

Hence $\grave{m}_2(\lambda)\grave{m}_2(\mu)\{\Delta_\lambda, \Delta_\mu\} = 0$ for any $\lambda, \mu \in \mathbb{C}^*$ with $\lambda \neq \pm\mu$. By the considerations at the beginning of the proof, the latter identity also holds for $\lambda = \pm\mu$. Since $\grave{m}_2(\lambda)$, Δ_λ are analytic and $\grave{m}_2(\lambda)$ does not vanish identically it then follows that $\{\Delta_\lambda, \Delta_\mu\} = 0$ on H_c^1 for any $\lambda, \mu \in \mathbb{C}^*$ as claimed. □

Remark 7.5. Using Lemma 5.13 one can show, arguing as in the proof of Theorem 7.4, that for any periodic eigenvalue $\lambda^{(1)}(v)$, $\lambda^{(2)}(v)$ of $Q(v)$ which are simple on a given open neighborhood V one has

$$\{\lambda^{(1)}, \lambda^{(2)}\} = 0, \quad \{\lambda^{(1)}, \Delta_\lambda\} = 0, \quad \lambda \in \mathbb{C}^*.$$

Using the asymptotics of Theorem 2.20 and Theorem 7.4 one concludes that Δ_λ Poisson commutes with H_{sinh}. Recall that by (2.50) and (2.51)

$$H_{\text{sinh}}(q,p) = -4(H_1(q,p) + H_1(-q,p)), \quad H_*(q,p) = -4(H_1(q,p) - H_1(-q,p)).$$

***Corollary* 7.6** *For any $v = (q,p) \in H^1_c$ and $\lambda \in \mathbb{C}^*$ one has,*

$$\{\Delta_\lambda, H_1\} = 0 \quad \text{and} \quad \{\Delta_\lambda, \tilde{H}_1\} = 0, \quad \tilde{H}_1(q,p) := H_1(-q,p)$$

and hence

$$\{\Delta_\lambda, H_{\text{sinh}}\} = 0, \quad \{\Delta_\lambda, H_*\} = 0$$

Proof. First we prove that for any $\lambda \in \mathbb{C}^*$, $\{\Delta_\lambda, H_1\} = 0$ on H^3_c. For $v \in H^3_c$ the asymptotics of Theorem 2.20 yield

$$\Delta(\mu, v) = \cosh(\sigma_1(\mu, v)) + O(\mu^{-2}) \quad \text{as} \quad |\mu| \to \infty$$

where

$$\sigma_1(\mu, v) = -\mathrm{i}\mu + -\mathrm{i}\frac{H_1(v)}{2\mu}.$$

Hence $\{\Delta_\lambda, \sigma_1(\mu)\} = \frac{1}{2\mathrm{i}\mu}\{\Delta_\lambda, H_1\}$. Furthermore by the Leibniz rule

$$\{\Delta_\lambda, \Delta_\mu\} = \sinh(\sigma_1(\mu))\{\Delta_\lambda, \sigma_1(\mu)\} + O(\mu^{-2}).$$

Since by Theorem 7.4 $\{\Delta_\lambda, \Delta_\mu\} = 0$, it then follows that

$$\sinh(\sigma_1(\mu))\frac{1}{2\mathrm{i}\mu}\{\Delta_\lambda, H_1\} + O(\mu^{-2}) = 0$$

and hence

$$\{\Delta_\lambda, H_1\} = 0.$$

Since Δ_λ and H_1 are analytic on H^1_c, the identity actually holds on H^1_c.

Finally since by Lemma 2.14(ii), $\Delta((16\lambda)^{-1}, q, p) = \Delta(\lambda, -q, p)$ it follows that with $\tilde{H}_1(q,p) = H_1(-q,p)$

$$\{\Delta_{(16\lambda)^{-1}}, \tilde{H}_1\}(q,p) = -\{\Delta_\lambda, H_1\}(-q,p) = 0.$$

$\square$

Lemma 7.1 can also be used to prove that the Dirichlet eigenvalues commute. Recall that on $\hat{W}$, defined by (6.4), the Dirichlet eigenvalues μ_n, $n \in \mathbb{Z}$, of $Q(v)$ are all simple, implying that $\dot{\chi}_D(\mu_n) \neq 0$.

***Lemma* 7.7** *The L^2-gradient $\partial_v \mu_n$ of μ_n is in L^2_c for any $v \in \hat{W}$ and $n \in \mathbb{Z}$. Hence for any $n, k \in \mathbb{Z}$, the Poisson bracket $\{\mu_n,\ \mu_k\}$ is well defined on $\hat{W}$ and vanishes there,*

$$\{\mu_n, \mu_k\} = 0.$$

Proof. Let $v \in \hat{W}$. It suffices to consider the case $n \neq k$. By Lemma 5.6, the L^2- gradient $\partial_v \mu_n$, $n \in \mathbb{Z}$, is of the form

$$\partial_v \mu_n = \frac{\grave{m}_1(\mu_n)}{\dot{\chi}_D(\mu_n)}(g_{n,1}, g_{n,2}P), \quad g_n := (g_{n,1}, g_{n,2}) = [\![M_2]\!]^{\sim}_{q,\mu_n} \tag{7.13}$$

where

$$g_{n,1}(x) = \frac{\mu_n}{2}(m_4^2(x,\mu_n) - m_2^2(x,\mu_n)) + \frac{1}{32\mu_n}(m_4^2(x,\mu_n)e^{q(x)} - m_2^2(x,\mu_n)e^{-q(x)})$$

and

$$g_{n,2}(x) = -\frac{1}{2}m_2(x,\mu_n)m_4(x,\mu_n).$$

Hence $\{\mu_n,\ \mu_k\}$ is well defined on $\hat{W}$ and

$$\{\mu_n, \mu_k\} = \frac{\grave{m}_1(\mu_n)}{\dot{\chi}_D(\mu_n)}\frac{\grave{m}_1(\mu_k)}{\dot{\chi}_D(\mu_k)}\big[\, [\![M_2]\!]^{\sim}_{q,\mu_n}, [\![M_2]\!]^{\sim}_{q,\mu_k} \big]_1.$$

By Lemma 7.1, one has

$$\begin{aligned}16\big[\,[\![M_2]\!]^{\sim}_{q,\mu_n},[\![M_2]\!]^{\sim}_{q,\mu_k}\,\big]_1 =&-\frac{\mu_n+\mu_k}{\mu_n-\mu_k}\big(M_2(x,\mu_n)\cdot JM_2(x,\mu_k)\big)^2\big|_0^1\\&+\frac{\mu_n-\mu_k}{\mu_n+\mu_k}\big(M_2(x,\mu_n)\cdot ZM_2(x,\mu_k)\big)^2\big|_0^1\\=&0\end{aligned}$$

and hence $\{\mu_n,\mu_k\}=0$ as claimed. Since $m_2(0,\mu_n)=0$ and $m_2(1,\mu_n)=0$, it follows that $g_{n,2}\in H^1_{\mathbb{C}}$ and hence $P(g_{n,2})\in L^2_{\mathbb{C}}$. It implies that $\partial_v\mu_n=\frac{\grave{m}_1(\mu_n)}{\dot{\chi}_D(\mu_n)}(g_{n,1},\ P(g_{n,2}))\in L^2_c$. □

We now compute the Poisson bracket of Δ_λ, $\lambda\in\mathbb{C}^*$, with the Dirichlet eigenvalue μ_n, $n\in\mathbb{Z}$, on $\hat{W}$, which will be used in the subsequent chapter.

Lemma 7.8 *For any $n\in\mathbb{Z}$ and $\lambda\in\mathbb{C}^*$, the Poisson bracket $\{\mu_n,\ \Delta_\lambda\}$ is well defined on $\hat{W}$. If $\lambda\neq\pm\mu_n$, then*

$$4\{\mu_n,\Delta_\lambda\}=\frac{\mu_n\delta(\mu_n)}{\dot{\chi}_D(\mu_n)}\frac{\lambda\chi_D(\lambda)}{\lambda^2-\mu_n^2}$$

whereas in the case $\lambda=\pm\mu_n$ one has

$$8\{\mu_n,\Delta_\lambda\}\big|_{\lambda=\pm\mu_n}=\mu_n\delta(\mu_n).$$

Proof. We argue as in the proof of Theorem 7.4 and write $\partial_v\Delta_\lambda$ as above in the form $\partial_v\Delta_\lambda=(f_{\lambda,1},f_{\lambda,2}P)$ where $f_\lambda=(f_{\lambda,1},f_{\lambda,2})$ is in $H^1_{\mathbb{C}}$, given by (7.10)- (7.11). Hence the Poisson bracket $\{\mu_n,\ \Delta_\lambda\}$ is well defined. By (7.13) it then follows that

$$\{\mu_n,\Delta_\lambda\}=\frac{\grave{m}_1(\mu_n)}{\dot{\chi}_D(\mu_n)}\big[\,[\![M_2]\!]_{q,\mu_n},f_\lambda\big]_1.$$

Note that by the Wronskian identity, $\grave{m}_1(\mu_n)\grave{m}_4(\mu_n)=1$ and hence $\grave{m}_1(\mu_n)\neq0$. Since by Lemma 7.3,

$$2\grave{m}_2(\lambda)f_\lambda=[\![\grave{m}_2(\lambda)M_1-\delta_\lambda M_2]\!]^{\sim}_{q,\lambda}-(\Delta_\lambda^2-1)\,[\![M_2]\!]^{\sim}_{q,\lambda}$$

one obtains

$$\frac{32\grave{m}_2(\lambda)\dot{\chi}_D(\mu_n)}{\grave{m}_1(\mu_n)}\{\mu_n,\Delta_\lambda\}=I_{n,\lambda}-(\Delta_\lambda^2-1)II_{n,\lambda}\tag{7.14}$$

where

$$\begin{aligned}I_{n,\lambda}:=&16\big[\,[\![M_2]\!]^{\sim}_{q,\mu_n},[\![\grave{m}_2(\lambda)M_1-\delta_\lambda M_2]\!]^{\sim}_{q,\lambda}\,\big]_1\\II_{n,\lambda}:=&16\big[\,[\![M_2]\!]^{\sim}_{q,\mu_n},[\![M_2]\!]^{\sim}_{q,\lambda}\,\big]_1\end{aligned}$$

By Lemma 7.1 one has for $\lambda\neq\pm\mu_n$

$$\begin{aligned}I_{n,\lambda}=&\frac{\lambda+\mu_n}{\lambda-\mu_n}\Big(M_2(x,\mu_n)\cdot J\big(\grave{m}_2(\lambda)M_1(x,\lambda)-\delta_\lambda M_2(x,\lambda)\big)\Big)^2\Big|_0^1\\&+\frac{\lambda-\mu_n}{\lambda+\mu_n}\Big(M_2(x,\mu_n)\cdot Z\big(\grave{m}_2(\lambda)M_1(x,\lambda)-\delta_\lambda M_2(x,\lambda)\big)\Big)^2\Big|_0^1\\=&\frac{-4\lambda\mu_n}{\lambda^2-\mu_n^2}\grave{m}_2(\lambda)^2\big(\grave{m}_4(\mu_n)^2\Delta_\lambda^2-1\big)\end{aligned}$$

and

$$\begin{aligned}II_{n,\lambda}=&-\frac{\lambda+\mu_n}{\lambda-\mu_n}\Big(M_2(x,\mu_n)\cdot JM_2(x,\lambda)\Big)^2\Big|_0^1+\frac{\lambda-\mu_n}{\lambda+\mu_n}\Big(M_2(x,\mu_n)\cdot ZM_2(x,\lambda)\Big)^2\Big|_0^1\\=&\frac{-4\lambda\mu_n}{\lambda^2-\mu_n^2}\grave{m}_4(\mu_n)^2\grave{m}_2(\lambda)^2.\end{aligned}$$

The identity (7.14) then becomes

$$\begin{aligned}\frac{32\grave{m}_2(\lambda)\dot{\chi}_D(\mu_n)}{\grave{m}_1(\mu_n)}\{\mu_n,\Delta_\lambda\}=&\frac{4\lambda\mu_n}{\lambda^2-\mu_n^2}\grave{m}_2(\lambda)^2\big(1-\grave{m}_4(\mu_n)^2\Delta_\lambda^2+(\Delta_\lambda^2-1)\grave{m}_4(\mu_n)^2\big)\\=&-\frac{4\lambda\mu_n}{\lambda^2-\mu_n^2}\grave{m}_2(\lambda)^2\big(\grave{m}_4(\mu_n)^2\Delta_\lambda^2-1-(\Delta_\lambda^2-1)\grave{m}_4(\mu_n)^2\big).\end{aligned}$$

Using that
$$1 - \grave{m}_4(\mu_n)^2 = \grave{m}_1(\mu_n)\grave{m}_4(\mu_n) - \grave{m}_4(\mu_n)^2 = \grave{m}_4(\mu_n)2\delta(\mu_n)$$
and multiplying the above identity by $\grave{m}_1(\mu_n)/8\grave{m}_2(\lambda)\dot{\chi}_D(\mu_n)$ one obtains in the case $\grave{m}_2(\lambda)(=\chi_D(\lambda)) \neq 0$, and hence $\lambda \neq \pm\mu_n$,
$$4\{\mu_n, \Delta_\lambda\} = \frac{\lambda\mu_n}{\lambda^2 - \mu_n^2}\frac{\grave{m}_2(\lambda)}{\dot{\chi}_D(\mu_n)}\grave{m}_1(\mu_n)\grave{m}_4(\mu_n)\delta(\mu_n) = \frac{\lambda\chi_D(\lambda)}{\lambda^2 - \mu_n^2}\frac{\mu_n\delta(\mu_n)}{\dot{\chi}_D(\mu_n)}.$$
By continuity, a remains valid for $\chi_D(\lambda) = 0$ with $\lambda \neq \pm\mu_n$. For the values $\lambda = \pm\mu_n$ it follows from
$$\lim_{\lambda\to\pm\mu_n}\frac{\lambda\chi_D(\lambda)}{\lambda^2 - \mu_n^2} = \frac{1}{2}\lim_{\lambda\to\pm\mu_n}\frac{\chi_D(\lambda)}{\lambda \mp \mu_n} = \frac{1}{2}\dot{\chi}_D(\mu_n)$$
that
$$4\{\mu_n, \Delta_\lambda\}\big|_{\lambda=\pm\mu_n} = \frac{\mu_n\delta(\mu_n)}{\dot{\chi}_D(\mu_n)}\frac{\dot{\chi}_D(\pm\mu_n)}{2}.$$
Recall that by Lemma 3.2(i), $\chi_D(-\lambda) = -\chi_D(\lambda)$ and hence $\dot{\chi}_D(-\lambda) = \dot{\chi}_D(\lambda)$ implying that
$$4\{\mu_n, \Delta_\lambda\}\big|_{\lambda=\pm\mu_n} = \frac{\mu_n\delta(\mu_n)}{2}.$$
□

Lemma 7.8 together with the derivation property of the Poisson bracket leads to the following

***Lemma* 7.9** *For any $k, n \in \mathbb{Z}$ and $\lambda_k \in \{\lambda_k^-, \lambda_k^+\}$, the Poisson bracket $\{\mu_n, \lambda_k\}$ is well defined on $H_r^1\backslash Z_k$ and, in the case $\lambda_k \notin \{\pm\mu_n\}$*
$$\{\mu_n, \lambda_k\} = -\frac{1}{4\dot{\Delta}_{\lambda_k}}\frac{\mu_n\delta(\mu_n)}{\dot{\chi}_D(\mu_n)}\frac{\lambda_k\chi_D(\lambda_k)}{\lambda_k^2 - \mu_n^2}.$$
In the case $\lambda_k \in \{\pm\mu_n\}$, χ_D vanishes at λ_k and the latter formula becomes by de L'Hospital's rule
$$\{\mu_n, \lambda_k\} = -\frac{1}{8\dot{\Delta}_{\lambda_k}}\mu_n\delta(\mu_n).$$

Remark 7.10. Note that since λ_k is simple $\lambda_k \neq \dot{\lambda}_k$ and hence $\dot{\Delta}(\lambda_k) \neq 0$.

Proof. By Lemma 5.13, $\lambda_k \in \{\lambda_k^+, \lambda_k^-\}$ is real analytic on $H_r^1 \setminus Z_k$ and $\partial\lambda_k = -\frac{\partial\Delta}{\dot{\Delta}}\big|_{\lambda_k}$. Since $\partial\mu_n \in L_r^2$ by Lemma 7.7, the Poisson bracket $\{\mu_n,\ \lambda_k\}$ is well defined on $H_r^1 \setminus Z_k$ and given by
$$\{\mu_n,\ \lambda_k\} = -\langle\partial\mu_n,\ JP^{-1}\partial\lambda_k\rangle_r = \langle\partial\mu_n,\ JP^{-1}\frac{\partial\Delta_\lambda}{\dot{\Delta}}\rangle_r\big|_{\lambda_k} = -\frac{1}{\dot{\Delta}(\lambda_k)}\{\mu_n,\ \Delta_\lambda\}\Big|_{\lambda_k}.$$
By Lemma 7.8,
$$4\{\mu_n,\ \lambda_k\} = -\frac{\mu_n\delta(\mu_n)}{\dot{\chi}_D(\mu_n)}\frac{1}{\dot{\Delta}(\lambda_k)}\frac{\lambda_k\chi_D(\lambda_k)}{\lambda_k^2 - \mu_n^2}.$$
□

7.2 Isospectral sets

In this section we describe properties of isospectral sets. For any given $v_0 \in H_r^1$, the isospectral set $Iso(v_0)$ is defined by
$$\mathrm{Iso}(v_0) = \{\ v \in H_r^1\ :\ \Delta_\lambda(v) = \Delta_\lambda(v_0)\ \ \forall\ \lambda \in \mathbb{C}^*\ \}.$$
Recall that for $q \in H_{\mathbb{C}}^1$, $\|q\|_1$ denotes the H^1-norm of q (cf (2.21)) and $\|q\|_{L^\infty}$ its sup norm, $\|q\|_{L^\infty} = \sup_{0\le x\le 1}|q(x)|$.

***Proposition* 7.11** *For any $v_0 = (q_0, p_0) \in H_r^1$ the following holds:*

(i) *For any* $v = (q,p) \in \mathrm{Iso}(v_0)$,

$$\|q\|_1^2 + \|p\|_1^2 \le \|q_0\|_1^2 + \|p_0\|_1^2 + 2(e^{\|q_0\|_{L^\infty}} - 1).$$

In particular , $\mathrm{Iso}(0) = \{0\}$.

(ii) $\mathrm{Iso}(v_0)$ *is compact.*

Proof. (i) Since for $(q,p) \in \mathrm{Iso}(q_0,p_0)$, $\Delta_\lambda(q,p) = \Delta_\lambda(q_0,p_0)$ for any $\lambda \in \mathbb{C}^*$, it follows from Lemma 2.14 that $\Delta_{1/16\lambda}(-q,p) = \Delta_{1/16\lambda}(-q_0,p_0)$ for any $\lambda \in \mathbb{C}^*$. Theorem 2.20 then yields

$$H_1(q,p) = H_1(q_0,p_0), \quad H_1(-q,p) = H_1(-q_0,p_0).$$

Since by (2.50), $H_{\sinh}(q,p) = -4(H_1(q,p) + H_1(-q,p))$ one concludes that $H_{\sinh}(q,p) = H_{\sinh}(q_0,p_0)$. The latter identity can be written as

$$\frac{1}{2}\|q\|_1^2 + \frac{1}{2}\|p\|_1^2 + \int_0^1 \big(\cosh(q) - (1+\frac{1}{2}q^2)\big)\,\mathrm{d}x = \frac{1}{2}\|q_0\|_1^2 + \frac{1}{2}\|p_0\|_1^2 + \int_0^1 \big(\cosh(q_0) - (1+\frac{1}{2}q_0^2)\big)\,\mathrm{d}x. \tag{7.15}$$

Note that $\cosh(x) - (1+\frac{1}{2}x^2) = \sum_{k\ge 2} \frac{1}{(2k)!}x^{2k} \ge 0$ and $0 \le \cosh(x) - 1 \le e^{|x|} - 1$ for any $x \in \mathbb{R}$ implying the claimed estimate

$$\|q\|_1^2 + \|p\|_1^2 \le \|q_0\|_1^2 + \|p_0\|_1^2 + 2(e^{\|q_0\|_{L^\infty}} - 1).$$

(ii) To prove that $Iso(v_0)$ is compact, let $v_n = (q_n,p_n)$, $n \ge 1$, be an arbitrary sequence in $\mathrm{Iso}(v_0)$. By item (i), $(v_n)_{n\ge1}$ is bounded in H_r^1, hence w.l.o.g. we assume that $(v_n)_{n\ge1}$ converges weakly in H_r^1 to an element $v_\infty = (q_\infty, p_\infty) \in H_r^1$. It remains to show that $v_\infty \in \mathrm{Iso}(v_0)$ and $(v_n)_{n\ge1}$ converges to v_∞ in H_r^1. Since by Corollary 2.15, Δ_λ is compact on H_r^1, $\Delta_\lambda(v_n) \to \Delta_\lambda(v_\infty)$ as $n \to \infty$ for any $\lambda \in \mathbb{C}^*$ and using that $\Delta_\lambda(v_n) = \Delta_\lambda(v_0)$, $n \in \mathbb{Z}$, one concludes that $\Delta_\lambda(v_\infty) = \Delta_\lambda(v_0)$ for any $\lambda \in \mathbb{C}^*$. It means that $v_\infty \in \mathrm{Iso}(v_0)$. Since by Rellich's theorem $q_n \to q_\infty$ strongly in L_r^2 and furthermore $(\|q_n\|_{L^\infty})_{n\ge1}$ is bounded it follows from (7.15) that $\|q_\infty\|_1^2 + \|p_\infty\|_1^2 = \lim_{n\to\infty}(\|q_n\|_1^2 + \|p_n\|_1^2)$ implying together with the weak convergence of $(v_n)_{n\ge1}$ in H_r^1 that $\lim_{n\to\infty} v_n = v_\infty$ in H_r^1. □

Remark 7.12. Similarly one can show that for any $v_0 \in H_r^2$, $Iso(v_0) \cap H_r^2$ is compact in H_r^2. Indeed, by (2.48)

$$H_3(q,p) = \int_0^1 \big((\psi')^2 + \psi^4\big)\,\mathrm{d}x + R_3(q,p), \quad \psi = \frac{1}{4}(Pp + q')$$

where

$$R_3(q,p) = \int_0^1 \Big(\frac{1}{4}\psi q' \cosh(q) + \frac{1}{4}\psi^2\cosh(q) + (\frac{1}{8}\sinh(q))^2\Big)\,\mathrm{d}x$$

Hence

$$H_3(q,p) + H_3(-q,p) = \int_0^1 \Big(\frac{1}{8}(Pp')^2 + \frac{1}{8}(q')^2 + (\frac{1}{4}Pp)^4 + 6(\frac{1}{4}Pp)^2(\frac{1}{4}q')^2 + (\frac{1}{4}q')^4\Big)\,\mathrm{d}x + R_3(q,p) + R_3(-q,p). \tag{7.16}$$

Arguing as in the proof of item (ii) of Proposition 7.11 it follows that $H_3(q,p) + H_3(-q,p)$ is a spectral invariant. Since $H_1(q,p) + H_1(-q,p)$ is also a spectral invariant and by (7.15) $\|q\|_1$ and $\|p\|_1$ are bounded on $Iso(v_0)$ one obtains the following bounds

$$\Big|\int_0^1 \psi q' \cosh(q)\,\mathrm{d}x\Big| \le e^{\|q\|_{L^\infty}} \|\psi\|_{L^2}\|q'\|_{L^2} \le Ce^{\|q\|_{L^\infty}}\big(\|q\|_1 + \|p\|_1\big)\|q\|_1$$

$$0 \le \int_0^1 \frac{1}{4}\psi^2\cosh(q)\,\mathrm{d}x \le e^{\|q\|_{L^\infty}} \int_0^1 \psi^2\,\mathrm{d}x \le Ce^{\|q\|_{L^\infty}}(\|q\|_1 + \|p\|_1)^2$$

and

$$\int_0^1 (\frac{1}{8}\sinh(q))^2\,\mathrm{d}x \le e^{\|q\|_{L^\infty}^2}.$$

This shows that $R(q,p) + R(-q,p)$ is bounded on $Iso(v_0)$. It then follows from (7.16) that $\|p\|_2 + \|q\|_2$ is bounded on $Iso(v_0) \cap H_r^2$. As in the proof of item (ii) of Proposition 7.11 one sees that any sequence $(v_n)_{n\ge1}$ in $Iso(v_0) \cap H_r^2$ has a weakly convergent subsequence in H_r^2, again denoted by $(v_n)_{n\ge1}$, and that its limit $v_\infty = (q_\infty, p_\infty)$ is in $Iso(v_0) \cap H^2_r$. Using that $H_3(q_n, p_n) + H_3(-q_n, p_n)$ is a spectral invariant and that due to the Sobolev embedding theorem $\lim_{n\to\infty} R(\pm q_n, p_n) = R(\pm q_\infty, p_\infty)$ and

$$\lim_{n\to\infty} \int_0^1 \psi(\pm q_n, p_n)^4 \, \mathrm{d}x = \int_0^1 \psi(\pm q_\infty, p_\infty)^4 \, \mathrm{d}x$$

it then follows that

$$\lim_{n\to\infty} \int_0^1 (Pp_n')^2 + (q_n'')^2 \, \mathrm{d}x = \int_0^1 (Pp_\infty')^2 + (q_\infty'')^2 \, \mathrm{d}x.$$

This together with the weak convergence of $(v_n)_{n\geq 1}$ in H^2_r then implies that $v_n \to v_\infty$ strongly in H^2_r. Using similar arguments we expect, but have not verified, that for any $(q_0, p_0) \in H^N_r$ with $N \geq 3$, $Iso(v_0) \cap H^N_r$ is compact in H^N_r.

To further investigate the isospectral sets we introduce vector fields whose flows leave the the isospectral sets invariant and move a single Dirichlet eigenvalue while leaving all other fixed. Consider for $v = (q, p) \in H^1_r$ the Hamiltonian vector field of the discriminant. Since $\partial_v \Delta = (f_{\lambda,1}, Pf_{\lambda,2})$ with $f_{\lambda,1}, f_{\lambda,2}$ defined by (7.10)-(7.11), it is given by

$$X_{\Delta_\lambda} = (-P^{-1}Pf_{\lambda,2}, P^{-1}f_{\lambda,1}) = (-f_{\lambda,2}, P^{-1}f_{\lambda,1}).$$

It then follows from Lemma 7.2 that $X_{\Delta_\lambda} \in H^1_r$. Furthermore by Theorem 2.2, X_{Δ_λ} depends analytically on λ and $v \in H^1_r$. Define for any $n \in \mathbb{Z}$, the vector field X_n on H^1_r by

$$X_n(v) := X_{\Delta_\lambda}(v)\Big|_{\lambda=\mu_n(v)}.$$

Since μ_n is real analytic on H^1_r, $X_n : H^1_r \to H^1_r$ is a real analytic vector field. Note however that it is not Hamiltonian. It follows that the initial value problem $\partial_t v = X_n(v)$, $v(0) = v_0$, has a unique local in time solution $t \mapsto v(t) \in H^1_r$ for any given initial date $v_0 \in H^1_r$. The Lie derivative $L_{X_n}(F)$ of a C^1 functional $F : H^1_r \to \mathbb{R}$ along a solution $t \mapsto v(t)$ of $\partial_t v = X_n(v)$ is given by

$$L_{X_n}(F) = \mathrm{d}F(X_n) = \{F, \Delta_\lambda\}\Big|_{\lambda=\mu_n}.$$

By Theorem 7.4, one has for any $\mu \in \mathbb{C}^*$,

$$L_{X_n}(\Delta_\mu) = \{\Delta_\mu, \Delta_\lambda\}\big|_{\lambda=\mu_n} = 0. \tag{7.17}$$

Hence X_n generates an isospectral flow, meaning that the solution $t \mapsto v(t)$ of $\partial_t v = X_n(v)$ evolves in $Iso(v_0)$. Since by Proposition 7.11(ii), $Iso(v_0)$ is compact, any solution $t \mapsto v(t)$ of $\partial_t v = X_n(v)$ exists for all values of time. Hence X_n admits a complete flow $X^t_n \equiv X^t_{X_n}$, $t \in \mathbb{R}$. Next we consider the motion of the Dirichlet eigenvalues along the flow X^t_n. By Lemma 7.8, for $m \neq n$

$$L_{X_n}(\mu_m) = \{\mu_m, \Delta_\lambda\}\Big|_{\lambda=\mu_n} = \frac{1}{8}\mu_n \delta(\mu_n)\delta_{mn}. \tag{7.18}$$

Lemma 7.13 *Assume that $v \in H^1_r$ with $\gamma_n(v) > 0$ for $n \in \mathbb{Z}$ given. Then along the flow $X^t_n(v)$, μ_n moves back and forth between $\lambda^-_n(v)$ and $\lambda^+_n(v)$ without stopping in the interior and bouncing off immediately at the end points, while for any $m \neq n$, μ_m is invariant under the flow.*

Proof. In view of (7.18), the discriminant Δ_λ and the Dirichlet eigenvalues μ_m, $m \neq n$, are invariant under the flow $X^t_n(v)$. Whereas the t-derivative of the function $\mu_n(t) := \mu_n(X^t_n(v))$ satisfies $\partial_t \mu_n = \mu_n \delta(\mu_n)/8$, where by Lemma 3.8, $\delta^2(\mu_n) = \Delta^2(\mu_n) - 1$. Hence for $X^t_n(v)$ with $\lambda^-_n < \mu_n(t) < \lambda^+_n$, $\delta(\mu_n(t)) \neq 0$ and hence $\partial_t \mu_n$ is either strictly increasing or strictly decreasing. Furthermore, one computes

$$8\partial_t^2 \mu_n = (\partial_t \mu_n)\delta(\mu_n) + \mu_n \partial_t \delta(\mu_n). \tag{7.19}$$

Since $\delta(\mu_n)\partial_t \delta(\mu_n) = \frac{1}{2}\partial_t \delta^2(\mu_n) = \frac{1}{2}\partial_t \Delta^2(\mu_n)$ and $\partial_t \Delta_\lambda(X^t_n(v)) = 0$ by (7.17), it follows that

$$\delta(\mu_n)\partial_t \delta(\mu_n) = \Delta(\mu_n)\dot{\Delta}(\mu_n)\partial_t \mu_n = \mu_n \Delta(\mu_n)\dot{\Delta}(\mu_n)\frac{\delta(\mu_n)}{8}.$$

Hence if $\delta(\mu_n) \neq 0$, $\partial_t \delta(\mu_n) = \mu_n \Delta(\mu_n)\dot{\Delta}(\mu_n)/8$. By a limiting argument, the assumption $\delta(\mu_n) \neq 0$ can be dropped and (7.19) becomes

$$64\partial_t^2 \mu_n = \mu_n \delta(\mu_n) + \mu_n^2 \Delta(\mu_n)\dot{\Delta}(\mu_n).$$

If $\mu_n(t) \in \{\lambda^\pm_n\}$, then $\delta(\mu_n(t)) = 0$, $\Delta(\mu_n(t)) = (-1)^n$, and $\dot{\Delta}(\mu_n(t)) \neq 0$ (since $\lambda^-_n < \lambda^+_n$) and it follows that $\partial_t^2 \mu_n \neq 0$. It means that μ_n bounces off immediately with $\partial_t \mu_n$ changing sign. □

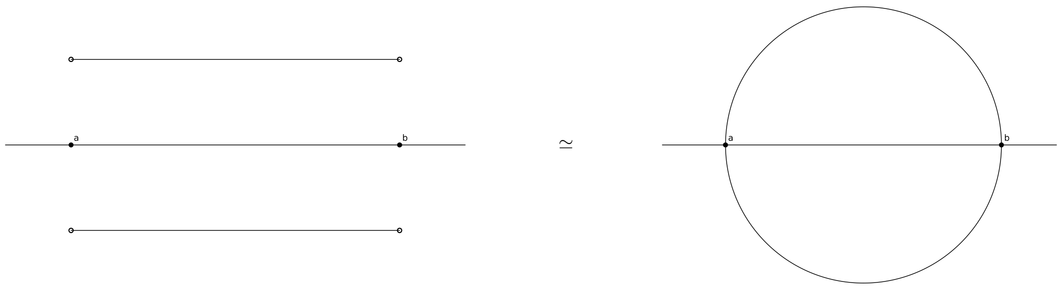

Figure 4: Circle over $[a, b]$

To describe the isospectral sets we define for any given interval $[a, b] \subset \mathbb{R}$ with $a < b$ the set

$$((a, b)) := \{(a, 0), (b, 0)\} \cup (a, b) \times \{1, -1\},$$

which reduces to a point if $a = b$. We endow $((a, b))$ with the topology that makes the map described in Figure 4 a homeomorphism between $((a, b))$ and a circle in $\mathbb{R}^2$ of radius $(b - a)/2$.

***Proposition* 7.14** *For any $v_0 \in H^1_r$, the map*

$$\mu \times \sigma : \mathrm{Iso}(v_0) \to \prod_{n\in\mathbb{Z}} ((\lambda^-_n, \lambda^+_n)), \quad v \mapsto \big(\mu_n(v), \mathrm{sign}(\delta(\mu_n(v))\big)_{n\in\mathbb{Z}},$$

is continuous and onto. Here the product $\prod_{n\in\mathbb{Z}} ((\lambda^-_n, \lambda^+_n))$ is endowed with the product topology and for any $x \in \mathbb{R}$, $\mathrm{sign}(x)$ is defined by

$$\mathrm{sign}(x) := \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases} .$$

Proof. By Lemma 6.1, the map $\mu \times \sigma$ is well-defined. Furthermore, for $n \in \mathbb{Z}$ arbitrary, any point in $((\lambda^-_n, \lambda^+_n))$ can be reached, by flowing along the flow X^t_n on $Iso(v_0)$. Since X^t_n affects only the n'th Dirichlet eigenvalue, one can use a finite combination of such flows to reach any combination of points on finitely many circles. Since by Proposition 7.11 $Iso(v_0)$ is compact in H^1_r one can construct a convergent sequence of potentials $(v_k)_{k\geq 1}$ in $Iso(v_0)$ so that for the limiting potential $v_\infty \in Iso(v_0)$, $\mu \times \sigma(v_\infty)$ takes on any prescribed element in $\prod_{n\in\mathbb{Z}} ((\lambda^-_n, \lambda^+_n))$. □

An immediate consequence of Proposition 7.14 and Proposition 7.11(ii) is the following result.

***Corollary* 7.15** *Any potential $v_0 \in H^1_r$ can be approximated by potentials $v \in \mathrm{Iso}(v_0)$ such that $\lambda^-_n(v_0) < \mu_n(v) < \lambda^+_n(v_0)$ for any $n \in \mathbb{Z}$ with $\gamma_n(v_0) > 0$.*

We finish this section by studying the translational flow. For any $v \in H^1_c$ and $t \in \mathbb{R}$ denote by v_t the translate of v, $v_t(x) = v(x + t)$. The vector field, corresponding to translation, is denoted by X_T and given by $X_T(v) = v'$. For any C^1-functional $F : H^1_c \to \mathbb{C}$, the Lie derivative of F in direction X_T is then given by

$$L_{X_T} F(v) = \partial_t F(v_t)\Big|_{t=0}.$$

Note that the fundamental solution $M(x, \lambda, v_t)$ is given by

$$M(x, \lambda, v_t) = M(x + t, \lambda, v)M^{-1}(t, \lambda, v),$$

where $M_x(t, \lambda, v) := M(x + t, \lambda, v)$ satisfies equation (2.3)

$$\partial_t M_x = J\left(\lambda - A(v_x) - B(v_x)^2/\lambda\right) M_x,$$

and the initial condition $M_x(0, \lambda, v) = M(x, \lambda, v)$. Hence it coincides with $M(t, \lambda, v_x)M(x, \lambda, v)$ and since v is one periodic, $v_1 = v$ and it follows that

$$\grave{M}(\lambda, v_t) = M(t, \lambda, v)\grave{M}(\lambda, v)M^{-1}(t, \lambda, v). \tag{7.20}$$

It implies that $\Delta_\lambda(v_t) = \Delta_\lambda(v)$ for any $\lambda \in \mathbb{C}^*$ and hence the flow $t \mapsto v_t$ is isospectral, i.e., it leaves the periodic spectrum of $Q(v)$ invariant. The following lemma says how the Dirichlet spectrum of $Q(v)$ evolves under this flow.

***Lemma* 7.16** *For any $n \in \mathbb{Z}$, one has on H^1_r*

$$L_{X_T}\mu_n = \frac{2\,\mathrm{sign}(\delta(\mu_n))\sqrt[+]{\Delta^2(\mu_n)-1}}{\dot{\chi}_D(\mu_n)}(\mu_n - \frac{e^{q(0)}}{16\mu_n}). \tag{7.21}$$

Proof. First we compute $L_{X_T}\grave{M}$. Since $\partial_t(M^{-1}(t,\lambda)M(t,\lambda)) = 0$ and $M(0,\lambda) = Id$ one has $\partial_t|_{t=0}M^{-1}(t,\lambda) = -\partial_t|_{t=0}M(t,\lambda)$. Hence by (7.20)

$$L_{X_T}\grave{M}(\lambda) = \partial_x M(0,\lambda)\grave{M}(\lambda) - \grave{M}(\lambda)\partial_x M(0,\lambda) = \Big[\partial_x M(0,\lambda), \grave{M}(\lambda)\Big].$$

By (2.3) it then follows that

$$\begin{aligned} L_{X_T}\grave{M}(\lambda) &= \Big[J\big(\lambda + \frac{1}{4}(Pp(0) + q'(0))Z\big) - \frac{1}{16\,\lambda}J\begin{pmatrix} e^{-q(0)} & \\ & e^{q(0)}\end{pmatrix}, \grave{M}(\lambda)\Big] \\ &= \Big[(\frac{1}{4}Pp(0) + q'(0))\begin{pmatrix} 1 & \\ & -1\end{pmatrix} + \begin{pmatrix} & \lambda - \frac{1}{16\,\lambda}e^{q(0)} \\ -\lambda + \frac{1}{16\,\lambda}e^{-q(0)} & \end{pmatrix}, \grave{M}(\lambda)\Big]. \end{aligned} \tag{7.22}$$

By the definition of μ_n, $\grave{m}_2(\mu_n) = 0$, and hence $\grave{M}(\mu_n) = \begin{pmatrix} \grave{m}_1(\mu_n) & 0 \\ \grave{m}_3(\mu_n) & \grave{m}_4(\mu_n)\end{pmatrix}$. It then follows that

$$0 = L_{X_T}\grave{m}_2(\mu_n) = \dot{\grave{m}}_2(\mu_n)L_{X_T}\mu_n + L_{X_T}\grave{m}_2(\lambda)\big|_{\lambda=\mu_n}$$

By (7.22) one has

$$\begin{aligned} L_{X_T}(\grave{m}_2(\lambda))\big|_{\lambda=\mu_n} &= \big(\mu_n - \frac{e^{q(0)}}{16\,\mu_n}\big)\big(\grave{m}_4(\mu_n) - \grave{m}_1(\mu_n)\big) \\ &= -\,(\mu_n - \frac{e^{q(0)}}{16\,\mu_n})2\delta(\mu_n) \end{aligned}$$

implying that, together with $\dot{\grave{m}}_2(\mu_n) = \dot{\chi}_D(\mu_n) \neq 0$

$$L_{X_T}(\mu_n) = \frac{2\delta(\mu_n)}{\dot{\chi}_D(\mu_n)}(\mu_n - \frac{e^{q(0)}}{16\,\mu_n}).$$

Since $\delta(\mu_n) = \mathrm{sign}(\delta(\mu_n))\sqrt[+]{\Delta^2(\mu_n)-1}$, the claimed formula (7.21) is proved. □

Remark 7.17. Note that $L_{X_T}\mu_n$ vanishes iff $\Delta^2(\mu_n) = 1$ or $16\mu_n^2 = e^{q(0)}$. Let us consider the case where $\Delta^2(\mu_n) = 1$ in more detail. It means that $\mu_n \in \{\lambda_n^+, \lambda_n^-\}$ and hence $\grave{m}_1(\mu_n) = (-1)^n$. Arguing as in the proof of Lemma 7.16 one obtains

$$\partial_t^2\big|_{t=0}\mu_n(v_t) = \frac{2\grave{m}_3(\mu_n)}{\dot{\chi}_D(\mu_n)}\big(\mu_n - \frac{e^{q(0)}}{16\,\mu_n}\big)^2.$$

If $\grave{m}_3(\mu_n) = 0$, then $\lambda_n^+ = \lambda_n^-$ and hence the n'th gap is closed and $\mu_n(v_t) = \mu_n(v)$ for any $t \in \mathbb{R}$. If $\grave{m}_3(\mu_n) \neq 0$, then, $\partial_t^2|_{t=0}\mu_n(v_t) \neq 0$ if $16\mu_n^2 \neq e^{q(0)}$.

8 Liouville coordinates

In this chapter we construct actions I_n, $n \in \mathbb{Z}$ on the complex neighborhood $\hat{W}$ of H^1_r in H^1_c, introduced in (6.4), and angles θ_n, $n \in \mathbb{Z}$, each on its appropriate domain of definition contained in $\hat{W}$.

8.1 Actions

First we introduce some more notation. For any $v \in \hat{W}$ and $m \in \mathbb{Z}$, denote by $\mathring{G}_m$ the sets, referred to as open gaps

$$\mathring{G}_m := \begin{cases} \{\, (1-\alpha)\lambda_m^- + \alpha\lambda_m^+ \,:\, 0 < \alpha < 1 \,\} & \text{if } \gamma_m \neq 0 \\ \emptyset & \text{if } \gamma_m = 0 \end{cases}.$$

In particular, for $v = 0$, $\mathring{G}_m = \emptyset \;\; \forall\, m \in \mathbb{Z}$. Furthermore, define for any $n \in \mathbb{Z}$ and $\lambda \in \mathbb{C}^* \setminus \bigcup_{m\in\mathbb{Z}}(\mathring{G}_m \cup -\mathring{G}_m)$ the function

$$F_n(\lambda) \equiv F_n(\lambda, v) := \int_{\lambda_n^+}^{\lambda} \frac{\dot{\Delta}(\mu, v)}{\sqrt[c]{\chi_p(\lambda, v)}}\,\mathrm{d}\mu, \quad F(\lambda) \equiv F_0(\lambda) \tag{8.1}$$

where the path of integration is chosen to be admissible, meaning that it does not intersect any open gap $\mathring{G}_m$, $m \in \mathbb{Z}$. To see that $F_n(\lambda)$ is well defined, we first need to establish some preliminary results. We begin by considering the case where $v = 0$.

***Lemma* 8.1** *For $v = 0$ the following holds:*

(i) For any $\lambda \in \mathbb{C}^*$,

$$\dot{\Delta}(\lambda)/\sqrt[c]{\chi_p(\lambda)} = -i\dot{\omega}(\lambda), \quad \omega(\lambda) = \lambda - 1/16\lambda.$$

Hence $\dot{\Delta}(\lambda)/\sqrt[c]{\chi_p(\lambda)}$ is analytic on $\mathbb{C}^*$ and

$$\int_{\Gamma^+} \dot{\Delta}(\lambda)/\sqrt[c]{\chi_p(\lambda)}\,\mathrm{d}\lambda = \int_{\Gamma^-} \dot{\Delta}(\lambda)/\sqrt[c]{\chi_p(\lambda)}\,\mathrm{d}\lambda$$

where Γ^+ and Γ^- denote the half circles $\lambda = \lambda_0^+ e^{\mathrm{i}\theta}$, $0 \le \theta \le \pi$, and, respectively $\lambda = \lambda_0^+ e^{-\mathrm{i}\theta}$, $0 \le \theta \le \pi$.

(ii) $F(\lambda) = -\mathrm{i}\omega(\lambda)$ is analytic on $\mathbb{C}^*$.

(iii) $F(\lambda_n^+) = -\mathrm{i}n\pi \;\; \forall\, n \in \mathbb{Z}$, hence for any $n \in \mathbb{Z}$, $F_n(\lambda) = F(\lambda) + \mathrm{i}n\pi \;\; \forall\, \lambda \in \mathbb{C}^*$.

(iv) For any $n \in \mathbb{Z}$, $(-1)^n\Delta(\lambda_n^+) = 1$, $F_n(\lambda_n^+) = 0$, and $\cosh(F_n(\lambda)) = (-1)^n\Delta(\lambda) \;\; \forall\, \lambda \in \mathbb{C}^*$. Hence $F_n(\lambda)$ is an analytic branch of $\cosh^{-1}((-1)^n\Delta(\lambda))$ on $\mathbb{C}^*$.

Proof. (i) Since by Lemma 2.16, $\Delta(\lambda) = \cos(\omega(\lambda))$ and by Lemma 6.20, $\sqrt[c]{\chi_p(\lambda)} = -\mathrm{i}\sin\omega(\lambda)$, $\dot{\Delta}(\lambda)/\sqrt[c]{\chi_p(\lambda)}$ is well defined on $\mathbb{C}^*$ and given by $-\mathrm{i}\dot{\omega}(\lambda) = -\mathrm{i}(1 + 1/16\lambda^2)$. In particular, it is analytic and since $\dot{\omega}(\lambda)$ is even in λ and $\lambda_0^+ = 1/4$ (Corollary 3.10) one has

$$\int_{\Gamma^+} \dot{\Delta}(\lambda)/\sqrt[c]{\chi_p(\lambda)}\,\mathrm{d}\lambda = \int_{\Gamma^-} \dot{\Delta}(\lambda)/\sqrt[c]{\chi_p(\lambda)}\,\mathrm{d}\lambda.$$

(ii) follows from item (i).
(iii) Since $\chi_p(\lambda) = -\sin^2(\omega(\lambda))$ one has $\omega(\lambda_n^\pm) = n\pi$ for any $n \in \mathbb{Z}$ and hence $F(\lambda_n^+) = -in\pi$, implying that for any $\lambda \in \mathbb{C}^*$, $F(\lambda) = F(\lambda_n^+) + F_n(\lambda)$.
(iv) Clearly for any $n \in \mathbb{Z}$, $(-1)^n\Delta(\lambda_n^+) = (-1)^n\cos\omega(\lambda_n^+) = 1$ and for any $n \in \mathbb{Z}$, $\lambda \in \mathbb{C}^*$ one has $\partial_\lambda \cosh((-1)^n\Delta(\lambda)) = \dot{\Delta}(\lambda)/\sqrt[c]{\chi_p(\lambda)}$. Since $F_n(\lambda_n^+) = 0$ and $\dot{F}_n(\lambda) = \dot{\Delta}(\lambda)/\sqrt[c]{\chi_p(\lambda)}$ as well, one concludes that $F_n(\lambda)$ is an analytic branch of $\cosh^{-1}((-1)^n\Delta(\lambda))$. □

Next let us consider the case where v is an element in $\hat{W}$. For any such v denote by $(U_n)_{n\in\mathbb{Z}}$, U_* a sequence of isolating neighborhoods (cf. Section 6.2) and choose counterclockwise oriented contours $(\Gamma_n)_{n\in\mathbb{Z}}$ with $\Gamma_n \subset U_n$ so that for any $n \in \mathbb{Z}$, Γ_n encircles the spectral gap $G_n \subset \mathbb{R}_{>0}$.

***Lemma* 8.2** *For any $v \in \hat{W}$ the following holds:*

(i) The function $\dot{\Delta}(\lambda)/\sqrt[c]{\chi_p(\lambda)}$, is analytic on $\mathbb{C}^* \setminus \bigcup_{\substack{m\in\mathbb{Z}\\ \gamma_m\neq 0}}(G_m \cup -G_m)$ and admits a product representation obtained from the product representation of $\dot{\Delta}(\lambda)$ (Lemma 6.11) and the one of $\sqrt[c]{\chi_p(\lambda)}$ (Lemma 6.18, 6.19).

(ii) For any $n \in \mathbb{Z}$ and any admissible path from λ_n^- to $\lambda \in U_n \setminus \mathring{G}_n$, $\int_{\lambda_n^-}^{\lambda} \frac{\dot{\Delta}(\mu)}{\sqrt[c]{\chi_p(\mu)}}\,\mathrm{d}\mu$ is well defined. It is analytic on $U_n \setminus G_n$ and satisfies $\int_{\lambda_n^-}^{\lambda_n^+} \frac{\dot{\Delta}(\mu)}{\sqrt[c]{\chi_p(\mu)}}\,\mathrm{d}\mu = 0$. In particular $\int_{\Gamma_n} \frac{\dot{\Delta}(\mu)}{\sqrt[c]{\chi_p(\mu)}}\,\mathrm{d}\mu = 0$.

(iii) The function $\dot\Delta(\lambda)/\sqrt[c]{\chi_p(\lambda)}$ *is even in* λ *on* $\mathbb{C}^* \setminus \bigcup_{\substack{m\in\mathbb{Z}\\ \gamma_m\neq 0}} (G_m \cup -G_m)$ *and*

$$\int_{\Gamma^+} \frac{\dot\Delta(\lambda)}{\sqrt[c]{\chi_p(\lambda)}}\,\mathrm{d}\lambda = \int_{\Gamma^-} \frac{\dot\Delta(\lambda)}{\sqrt[c]{\chi_p(\lambda)}}\,\mathrm{d}\lambda$$

where Γ^+ *and* Γ^- *denote the half circles* $\lambda = \lambda_0^+ e^{\mathrm{i}\theta}$ *and, respectively,* $\lambda = \lambda_0^+ e^{-\mathrm{i}\theta}$, $0 \le \theta \le \pi$.

(iv) For any $n \in \mathbb{Z}$,

$$\int_{\lambda_n^+}^{\lambda_{n+1}^-} \frac{\dot\Delta(\lambda)}{\sqrt[c]{\chi_p(\lambda)}}\,\mathrm{d}\lambda = -\mathrm{i}\pi.$$

Proof. In order to simplify notation, we only consider potentials v in H^1_r. The case where $v \in \hat W$ is proved in the same way. (i) Since $\sqrt[c]{\chi_p(\lambda)}$ does not vanish on $\mathbb{C}^* \setminus \bigcup_{n\in\mathbb{Z}}(G_n \cup -G_n)$ and is analytic there (Lemma 6.21) and since $\dot\Delta(\lambda)$ is analytic on $\mathbb{C}^*$ (Corollary 2.15) it follows that the quotient $\dot\Delta(\lambda)/\sqrt[c]{\chi_p(\lambda)}$ is well defined and analytic on $\mathbb{C}^* \setminus \bigcup_{n\in\mathbb{Z}}(G_n \cup -G_n)$. Furthermore, by Lemma 6.15, $\gamma_n = 0$ implies that $\dot\lambda_n = \tau_n$. In such a case the terms $(\dot\lambda_n - \lambda)/w_n(\lambda)$ and $(\dot\lambda_n + \lambda)/w_n(-\lambda)$ are both equal to 1 and one deduces from the product representation of $\dot\Delta(\lambda)$ and $\sqrt[c]{\chi_p(\lambda)}$ that $\dot\Delta(\lambda)/\sqrt[c]{\chi_p(\lambda)}$ extends analytically to $\lambda = \tau_n$
(ii) By (i) it suffices to consider $n \in \mathbb{Z}$ with $\gamma_n \neq 0$. Since the quotient $\dot\Delta(\lambda)/\sqrt[c]{\chi_p(\lambda)}$ has a singularity of the form $1/\sqrt{\lambda_n^\pm - \lambda}$ locally around $\lambda_n^\pm$, which is integrable, the integral $\int_{\lambda_n^-}^{\lambda} \dot\Delta(\lambda)/\sqrt[c]{\chi_p(\lambda)}\,\mathrm{d}\lambda$ is well defined for $\lambda \in U_n \setminus \mathring{G}_n$ along any admissible path in $U_n \setminus \mathring{G}_n$. To see that its value is independent of the admissible path chosen, consider $\int_{\Gamma_n} \dot\Delta(\lambda)/\sqrt[c]{\chi_p(\lambda)}\,\mathrm{d}\lambda$. By contour deformation and taking into account the definition of the canonical root and the fact that $(-1)^n\Delta(\lambda) \ge 1$ for $\lambda \in G_n$ and $(-1)^n\Delta(\lambda_n^\pm) = 1$ one gets

$$\int_{\Gamma_n} \frac{\dot\Delta(\lambda)}{\sqrt[c]{\chi_p(\lambda)}}\,\mathrm{d}\lambda = 2\int_{\lambda_n^-}^{\lambda_n^+} \frac{(-1)^{n+1}\dot\Delta(\lambda)}{\sqrt[+]{\chi_p(\lambda)}}\,\mathrm{d}\lambda = -2\cosh^{-1}((-1)^n\Delta(\lambda))\Big|_{\lambda_n^-}^{\lambda_n^+} = 0.$$

Hence for any admissible path in $U_n \setminus \mathring{G}_n$,

$$\int_{\lambda_n^-}^{\lambda_n^+} \frac{\dot\Delta(\lambda)}{\sqrt[c]{\chi_p(\lambda)}}\,\mathrm{d}\lambda = 0.$$

(iii) By Lemma 2.14(i), $\Delta(-\lambda) = \Delta(\lambda)$, hence $\dot\Delta(-\lambda) = -\dot\Delta(\lambda)$, and by Lemma 6.21(iii), $\sqrt[c]{\chi_p(-\lambda)} = -\sqrt[c]{\chi_p(\lambda)}$. Hence the quotient $\dot\Delta(\lambda)/\sqrt[c]{\chi_p(\lambda)}$ is an even function on $\mathbb{C}^* \setminus \bigcup_{\substack{m\in\mathbb{Z}\\ \gamma_m\neq 0}} (G_m \cup -G_m)$. Note that the half circles $\Gamma^+ = \{\ \lambda_0^+ e^{\mathrm{i}\theta} \ : \ 0 \le \theta \le \pi\ \}$ and $\Gamma^- = \{\ \lambda_0^+ e^{-\mathrm{i}\theta} \ : \ 0 \le \theta \le \pi\ \}$ both start at λ_0^+ and end at $-\lambda_0^+$, intersecting the real line only in these two points. Parametrizing Γ^- by $\lambda(\theta) = \lambda_0^+ e^{\mathrm{i}\theta}$ $0 \ge \theta \ge -\pi$, one gets

$$J := \int_{\Gamma^-} \frac{\dot\Delta(\lambda)}{\sqrt[c]{\chi_p(\lambda)}}\,\mathrm{d}\lambda = \int_0^{-\pi} \frac{\dot\Delta(\lambda(\theta))}{\sqrt[c]{\chi_p(\lambda(\theta))}}\mathrm{i}\lambda(\theta)\mathrm{d}\theta.$$

Making the change of variables $\alpha := \theta + \pi$ one has $\lambda(\theta) = \lambda(\alpha - \pi) = -\lambda(\alpha)$. It then follows from the eveness of $\dot\Delta(\lambda)/\sqrt[c]{\chi_p(\lambda)}$ that

$$J = \int_\pi^0 \frac{\dot\Delta(\lambda(\alpha))}{\sqrt[c]{\chi_p(\lambda(\alpha))}}(-\mathrm{i}\lambda(\alpha))\mathrm{d}\alpha = \int_{\Gamma^+} \frac{\dot\Delta(\lambda)}{\sqrt[c]{\chi_p(\lambda)}}\,\mathrm{d}\lambda$$

proving the claimed identity.
(iv) By the definition of the canonical root, $\mathrm{i}(-1)^n\sqrt[c]{\chi_p(\lambda)} = \sqrt[+]{1 - \Delta^2(\lambda)} > 0$ for any $\lambda_n^+ < \lambda < \lambda_{n+1}^-$, $n \in \mathbb{Z}$. Since $(-1)^k\Delta(\lambda_k^+) = (-1)^k\Delta(\lambda_k^-) = 1$ for any $k \in \mathbb{Z}$

$$\int_{\lambda_n^+}^{\lambda_{n+1}^-} \frac{\dot\Delta(\lambda)}{\sqrt[c]{\chi_p(\lambda)}}\,\mathrm{d}\lambda = \mathrm{i}\sin^{-1}((-1)^n\Delta(\lambda))\Big|_{\lambda_n^+}^{\lambda_{n+1}^-} = -\mathrm{i}\pi.$$

The case where $v \in \hat W$ is treated in the same fashion. □

In a straightforward way, Lemma 8.2 together with results we have established previously, lead to the following.

***Corollary* 8.3** *For any $n \in \mathbb{Z}$ and $v = (q,p) \in \hat{W}$ the following holds:*

(i) The function $F_n : \mathbb{C}^* \setminus \bigcup_{m\in\mathbb{Z}}(\mathring{G}_m \cup -\mathring{G}_m) \to \mathbb{C}$ is single valued and on $\mathbb{C}^* \setminus \bigcup_{\substack{m\in\mathbb{Z}\\ \gamma_m\neq 0}}(G_m \cup -G_m)$ an analytic branch of $\cosh^{-1}((-1)^n\Delta(\lambda))$.

(ii) For any $\lambda \in \mathbb{C}^* \setminus \bigcup_{m\in\mathbb{Z}}(\mathring{G}_m \cup -\mathring{G}_m)$,

$$F_n(\lambda) = F(\lambda) + \mathrm{i}n\pi, \quad F(-\lambda) = -F(\lambda) + F(-\lambda_0^+).$$

(iii) For any $\lambda \in \mathbb{C}^* \setminus \bigcup_{m\in\mathbb{Z}}(\mathring{G}_m \cup -\mathring{G}_m)$,

$$F(\lambda, q, p) = -F(1/16\lambda, -q, p).$$

(Here we used that $1/16\lambda \notin \pm\mathring{G}_m(-q,p)$ for any $m \in \mathbb{Z}$.)

(iv) For $\lambda \in U_n \setminus G_n$ sufficiently close to G_n,

$$F_n(\lambda) = \log(-1)^n\big(\Delta(\lambda) + \sqrt[c]{\chi_p(\lambda)}\big)$$

where log *denotes the principal branch of the logarithm.*

Remark 8.4. Since by Corollary 8.3(i), $\cosh F(\lambda) = \Delta(\lambda)$ is the trace of $M(1,\lambda)$ and by the Wronskian identity, $\det M(1,\lambda) = 1$, it follows that $F(\lambda)$ and $-F(\lambda)$ are the Floquet exponents of the Floquet matrix $M(1,\lambda)$.

Proof. In order to simplify notation we only consider potentials $v \in H_r^1$. The case where $v \in \hat{W}$ is proved in the same way. (i) By Lemma 8.2, F_n is a single valued function on $\mathbb{C}^* \setminus \bigcup_{m\in\mathbb{Z}}(\mathring{G}_m \cup -\mathring{G}_m)$ and analytic on $\mathbb{C}^* \setminus \bigcup_{\substack{m\in\mathbb{Z}\\ \gamma_m\neq 0}}(G_m \cup -G_m)$. Since $F_n(\lambda_n^+) = 0$, $(-1)^n\Delta(\lambda_n^+) = 1$, $\cosh^{-1}(z)$ is well defined mod $2\pi\mathrm{i}\mathbb{Z}$ and $\cosh^{-1}((-1)^n\Delta(\lambda))$ has a well defined derivative on $\mathbb{C}^* \setminus \bigcup_{\substack{m\in\mathbb{Z}\\ \gamma_m\neq 0}}(G_m \cup -G_m)$ given by $\frac{\dot{\Delta}(\lambda)}{\sqrt[c]{\chi_p(\lambda)}}$ it follows that $F_n(\lambda)$ is a branch of $\cosh^{-1}((-1)^n\Delta(\lambda))$.
(ii) It follows from Lemma 8.2(iv) that for any $n \neq 0$

$$F(\lambda) = (\int_{\lambda_0^+}^{\lambda_n^+} + \int_{\lambda_n^+}^{\lambda})\frac{\dot{\Delta}(\mu)}{\sqrt[c]{\chi_p(\mu)}}\,\mathrm{d}\mu = F(\lambda_n^+) + F_n(\lambda).$$

For any $n \geq 1$, write

$$F(\lambda_n^+) = \sum_{k=0}^{n-1}(\int_{\lambda_k^+}^{\lambda_{k+1}^-} + \int_{\lambda_{k+1}^-}^{\lambda_{k+1}^+})\frac{\dot{\Delta}(\mu)}{\sqrt[c]{\chi_p(\mu)}}\,\mathrm{d}\mu.$$

Hence by Lemma 8.2(ii) and (iv), $F(\lambda_n^+) = -\mathrm{i}n\pi$. Similarly one shows that for $n \leq -1$, one also has $F(\lambda_n^+) = -\mathrm{i}n\pi$. Furthermore,

$$F(-\lambda) = \int_{\lambda_0^+}^{-\lambda}\frac{\dot{\Delta}(\mu)}{\sqrt[c]{\chi_p(\mu)}}\,\mathrm{d}\mu = F(-\lambda_0^+) + \int_{-\lambda_0^+}^{-\lambda}\frac{\dot{\Delta}(\mu)}{\sqrt[c]{\chi_p(\mu)}}\,\mathrm{d}\mu.$$

Since by Lemma 8.2(ii), $\frac{\dot{\Delta}(\mu)}{\sqrt[c]{\chi_p(\mu)}}$ is even it follows that $\int_{-\lambda_0^+}^{-\lambda}\frac{\dot{\Delta}(\mu)}{\sqrt[c]{\chi_p(\mu)}}\,\mathrm{d}\mu = -F(\lambda)$. Altogether we thus have proved the claimed identity $F(-\lambda) = F(-\lambda_0^+) - F(\lambda)$.
(iii) It suffices to prove the claimed identity for $\lambda \in \mathbb{C} \setminus \mathbb{R}$. By the change of variable $\mu \mapsto \nu := 1/16\mu$,

$$F(1/16\lambda, -q, p) = \int_{\lambda_0^+(-q,p)}^{1/16\lambda}\frac{\dot{\Delta}(\mu,-q,p)}{\sqrt[c]{\chi_p(\mu,-q,p)}}\,\mathrm{d}\mu = \int_{(16\lambda_0^+(-q,p))^{-1}}^{\lambda}\frac{\dot{\Delta}(1/16\nu,-q,p)}{\sqrt[c]{\chi_p(1/16\nu,-q,p)}}\frac{-1}{16\nu^2}\,\mathrm{d}\nu.$$

Using that $(16\lambda_0^+(-q,p))^{-1} = \lambda_0^+(q,p)$ (Theorem 3.9), $\Delta(1/16\nu,-q,p) = \Delta(\nu,q,p)$ (Lemma 2.14(ii))), and $\sqrt[c]{\chi_p(1/16\nu,-q,p)} = -\sqrt[c]{\chi_p(\nu,q,p)}$ (Lemma 6.21(iii)) it then follows that

$$F(1/16\lambda,-q,p) = -\int_{\lambda_0^+(q,p)}^{\lambda}\frac{\dot{\Delta}(\nu,q,p)}{\sqrt[c]{\chi_p(\nu,q,p)}}\,\mathrm{d}\nu = -F(\lambda,q,p).$$

(iv) For $\lambda \in U_n \setminus G_n$ sufficiently close to G_n, $\mathrm{Re}\big((-1)^n(\Delta(\lambda) + \sqrt[c]{\chi_p(\lambda)})\big) > 0$ and hence $f_n(\lambda) := \log(-1)^n(\Delta(\lambda) + \sqrt[c]{\chi_p(\lambda)})$ is well defined. Since $f_n(\lambda_n^+) = 0$ and the derivatives of $f_n(\lambda)$ and $F_n(\lambda)$ for $\lambda \in U_n \setminus G_n$ coincide where $f_n(\lambda)$ is defined, it follows that for those values of λ, $f_n(\lambda) = F_n(\lambda)$. □

By the same arguments as in the case where $v \in H^1_r$ one can prove corresponding results for $F_n(\lambda, v)$ for v in the complex neighborhood $\hat{W}$ of H^1_r. Given any $v_0 = (q_0, p_0) \in H^1_r$, let $(U_n)_{n\in\mathbb{Z}}$, U_* be a sequence of isolating neighborhoods (cf Section 6.2) and Γ_n a sequence of counterclockwise oriented contours $(\Gamma_n)_{n\in\mathbb{Z}}$ with $\Gamma_n \subset U_n$ so that for any $n \in \mathbb{Z}$, Γ_n encircles the spectral gap G_n. Furthermore chose a neighborhood V_{v_0} of v_0 in $\hat{W}$ so that $(U_n)_{n\in\mathbb{Z}}$ and $(\Gamma_n)_{n\in\mathbb{Z}}$ work for any potential $v \in V_{v_0}$.

***Proposition* 8.5** *(i) For any $n \in \mathbb{Z}$ and $v_0 \in \hat{W}$, $F_n(\lambda, v)$ is analytic as a function of λ and v on $\big(\mathbb{C}^* \setminus \bigcup_{m\in\mathbb{Z}}(U_m \cup -U_m)\big) \times V_{v_0}$. The L^2-gradient $\partial F_n(\lambda, \cdot)$ of F_n with respect to $v = (q, p)$ is given by*

$$\partial F_n(\lambda, v) = \frac{\partial\Delta(\lambda, v)}{\sqrt[c]{\chi_p(\lambda, v)}}.$$

(ii) For $v_0 = 0$ and $n \in \mathbb{Z}$, $\partial F_n(\lambda, 0) = 0$.

Proof. (i) By Corollary 2.15 and Lemma 6.21

$$F_n : \big(\mathbb{C}^* \setminus \bigcup_{m\in\mathbb{Z}} (U_m \cup -U_m)\big) \times V_{v_0} \to \mathbb{C}$$

is analytic. To obtain the claimed formula for the L^2-gradient of F_n, we use that by Corollary 8.3(i), $F_n(\lambda, v)$ is an analytic branch of $\cosh^{-1}((-1)^n\Delta(\lambda, v))$. Since $\Delta(\lambda, v)$ is analytic on $\mathbb{C}^* \times H^1_c$ it then follows by the chain rule that

$$\partial F_n(\lambda, v) = \frac{\partial\Delta(\lambda, v)}{\sqrt[c]{\chi_p(\lambda, v)}}.$$

(ii) By Lemma 5.2(iii), $\partial\Delta(\lambda, v) = 0$ at $v = 0$ implying that $\partial F(\lambda, 0) = 0$. □

With these preparation we can now define our candidates for the actions. in fact it turns out to be useful to define k-level actions for any $k \in \mathbb{Z}$, which were first introduced in [18] in the context of the defocusing NLS equation (cf also [6]).

Given any $k \in \mathbb{Z}$, define for $n \in \mathbb{Z}$, $v \in \hat{W}$

$$J_{k,n}(v) := -\frac{1}{\pi}\int_{\Gamma_n} \frac{1}{\lambda}(4\lambda)^k F(\lambda, v)\,\mathrm{d}\lambda \tag{8.2}$$

where $\Gamma_n \subset U_n$ is the contour around G_n introduced above. Since $F(\cdot, v)$ is analytic on $\mathbb{C}^* \setminus \bigcup_{\substack{m\in\mathbb{Z}\\ \gamma_m\neq 0}} (G_m \cup -G_m)$ it follows from Cauchy's theorem that $J_{k,n}$ does not depend on the specific choice of Γ_n.

Integrating by parts one obtains

$$J_{0,n}(v) = \frac{1}{\pi}\int_{\Gamma_n} \log\lambda \frac{\dot{\Delta}(\lambda, v)}{\sqrt[c]{\chi_p(\lambda, v)}}\,\mathrm{d}\lambda \tag{8.3}$$

where we used that by (8.1) (cf also Proposition 8.5(i)) $\dot{F}(\lambda, v) = \dot{\Delta}(\lambda, v)/\sqrt[c]{\chi_p(\lambda, v)}$. Similarly, for $k \in \mathbb{Z} \setminus \{0\}$, one gets

$$J_{k,n}(v) = \frac{1}{\pi}\int_{\Gamma_n} \frac{1}{k}(4\lambda)^k \frac{\dot{\Delta}(\lambda, v)}{\sqrt[c]{\chi_p(\lambda, v)}}\,\mathrm{d}\lambda. \tag{8.4}$$

Recall from (6.4) that $\hat{W}$ is a neighborhood of H^1_r in H^1_c which is connected and has the property that for any $(q, p) \in \hat{W}$, also $(-q, p)$ is in $\hat{W}$. Further recall from Section 6.2 that $Z_n = \{\, v \in \hat{W} : \ \gamma_n(v) = 0 \,\}$.

***Proposition* 8.6** *(i) For any $k, n \in \mathbb{Z}$, $J_{k,n}$ is analytic on $\hat{W}$ with L^2-gradient given by*

$$\partial J_{k,n}(v) = -\frac{1}{\pi}\int_{\Gamma_n} \frac{1}{\lambda}(4\lambda)^k \frac{\partial\Delta(\lambda, v)}{\sqrt[c]{\chi_p(\lambda, v)}}\,\mathrm{d}\lambda.$$

(ii) For any $k, n \in \mathbb{Z}$, $v = (q, p) \in \hat{W}$

$$J_{k,n}(-q, p) = J_{-k,-n}(q, p).$$

(iii) For any $k, n \in \mathbb{Z}$, $J_{k,n}$ vanishes on Z_n. Furthermore the restriction of $J_{k,n}$ to H^1_r takes values in $\mathbb{R}_{\geq 0}$ and $J_{k,n}(v) = 0$ iff $\gamma_n(v) = 0$.

Proof. Item (i) follows from Proposition 8.5. Concerning item (ii) use the change of variable of integration, $\mu := \frac{1}{16\lambda}$. It is easy to check that the counterclockwise oriented curve $\Gamma_n(-q,p)$ is mapped by this change of variables to a counterclockwise oriented curve which, by Lemma 6.6(ii), encircles the spectral gap $G_{-n}(q,p)$. Hence

$$\begin{aligned} J_{k,n}(-q,p) &= -\frac{1}{\pi}\int_{\Gamma_n(-q,p)} \frac{1}{\lambda}(4\lambda)^k F(\lambda,-q,p)\,\mathrm{d}\lambda \\ &= -\frac{1}{\pi}\int_{\Gamma_{-n}(q,p)} 16\mu\left(\frac{1}{4\mu}\right)^k F(\frac{1}{16\mu},-q,p)(-\frac{1}{16\mu^2})\,\mathrm{d}\mu. \end{aligned}$$

Since by Corollary 8.3(iii), $F(\frac{1}{16\mu},-q,p) = -F(\mu,q,p)$ it then follows that

$$J_{k,n}(-q,p) = -\frac{1}{\pi}\int_{\Gamma_{-n}(q,p)} \frac{1}{\mu}(4\mu)^{-k}F(\mu,q,p)\,\mathrm{d}\mu = J_{-k,-n}(q,p).$$

(iii) Let $v \in \hat{W}$ and $n \in \mathbb{Z}$. If $\gamma_n = 0$, then by Corollary 8.3(i) the integrand of (8.2) is analytic in $\lambda \in U_n$ and hence by Cauchy's theorem, $J_{k,n}(v) = 0$ for any $k \in \mathbb{Z}$. Now assume that $v \in H^1_r$ and $\gamma_n \neq 0$. First we consider the case $k \neq 0$. Our starting point is formula (8.4). Since by Lemma 8.2(ii), $\int_{\Gamma_n} \frac{\dot{\Delta}(\lambda,v)}{\sqrt[c]{\chi_p(\lambda,v)}}\,\mathrm{d}\lambda = 0$, one has

$$J_{k,n}(v) = \frac{1}{\pi}\int_{\Gamma_n} \frac{1}{k}\big((4\lambda)^k - (4\dot{\lambda}_n)^k\big)\frac{\dot{\Delta}(\lambda,v)}{\sqrt[c]{\chi_p(\lambda,v)}}\,\mathrm{d}\lambda. \tag{8.5}$$

We then deform the contour Γ_n to G_n to get by (ii) of Lemma 6.22

$$J_{k,n}(v) = \frac{2}{\pi}\int_{\lambda_n^-}^{\lambda_n^+} \frac{1}{k}\big((4\lambda)^k - (4\dot{\lambda}_n)^k\big)\frac{(-1)^{n+1}\dot{\Delta}(\lambda,v)}{\sqrt[+]{\chi_p(\lambda,v)}}\,\mathrm{d}\lambda$$

implying that $J_{k,n}(v)$ is real valued. For $k \geq 1$ the product representation of $\dot{\Delta}$ (cf (6.27)) together with the fact that $\dot{\lambda}_n \in \mathbb{R}$ for all $n \in \mathbb{Z}$ and $\dot{\lambda}_* \in \mathrm{i}\mathbb{R}_{>0}$ (Lemma 6.2) yields

$$(-1)^{n+1}(\lambda - \dot{\lambda}_n)\dot{\Delta}(\lambda) > 0 \qquad \forall\, \lambda \in G_n \setminus \{\dot{\lambda}_n\}.$$

Since $(4\lambda)^k - (4\dot{\lambda}_n)^k = 4^k(\lambda - \dot{\lambda}_n)\sum_{l=0}^{k-1}\lambda^{k-1-l}\dot{\lambda}_n^l$ one then concludes that $J_{k,n}(v) > 0$. Item (ii) then implies that $J_{-k,n}(v) > 0$ for $k > 0$ as well. Finally, the case $k = 0$ is treated similarly: Using formula (8.3) as a starting point one obtains

$$J_{0,n}(v) = \frac{2}{\pi}\int_{\lambda_n^-}^{\lambda_n^+} (\log\lambda - \log\dot{\lambda}_n)\frac{(-1)^{n+1}\dot{\Delta}(\lambda,v)}{\sqrt[+]{\chi_p(\lambda,v)}}\,\mathrm{d}\lambda.$$

Note that $\log\frac{\lambda}{\dot{\lambda}_n}$ changes sign at $\lambda = \dot{\lambda}_n$, hence $(\log\frac{\lambda}{\dot{\lambda}_n})(-1)^{n+1}\dot{\Delta}(\lambda,v) > 0$ for any $\lambda \in G_n \setminus \{\dot{\lambda}_n\}$, implying that $J_{0,n}(v) > 0$. □

Similarly as in the case of the defocusing NLS equation we have the following

Lemma 8.7 *For any $k,n \in \mathbb{Z}$, $J_{k,n}$ is a compact function on H^1_r.*

Proof. By Proposition 2.5 and Proposition 2.6, $\Delta(\lambda,v)$ and $\dot{\Delta}(\lambda,v)$ are compact functions on H^1_r. By the same arguments as in the proof of ([6], Proposition 13.2) the claimed result follows. □

For $v \in \hat{W}$ and $n \in \mathbb{Z}$ define

$$I_n(v) := -\frac{4}{\pi}\int_{\Gamma_n} \frac{1}{\lambda}F(\lambda,v)\,\mathrm{d}\lambda. \tag{8.6}$$

Note that by integration by parts, $I_n(v) = 4J_{0,n}(v)$.

Theorem 8.8 *(i) After shrinking the complex neighborhood $\hat{W}$ of H^1_r if needed, each quotient $\tau_n I_n/\gamma_n^2$, $n \geq 0$, is analytic on $\hat{W}$ and locally uniformly of the form*

$$\frac{I_n\tau_n}{\gamma_n^2} = 1 + \ell_n^2. \tag{8.7}$$

The real part of $I_n\tau_n/\gamma_n^2$ is positive and locally uniformly bounded away from zero so that

$$\xi_n := \sqrt[+]{I_n\tau_n/\gamma_n^2},$$

is a well defined, real analytic, non-vanishing function on $\hat{W}$ satisfying $\xi_n = 1+\ell_n^2$ locally uniformly. At the zero potential, $\xi_n = \sqrt[+]{\frac{1}{2\tau_n}\sqrt[+]{4n^2\pi^2+1}}$ for all $n \geq 0$.

(ii) Locally uniformly on $\hat{W}$

$$J_{1,n} = \frac{4}{\pi}\int_{\Gamma_n}(\lambda - \dot{\lambda}_n)\dot{F}(\lambda)\,\mathrm{d}\lambda = \gamma_n^2(1+\ell_n^2) \quad \textit{as} \;\; n \to \infty.$$

Proof. (i) Let $n \geq 0$. Since I_n (Proposition 8.6) and τ_n, γ_n^2 (Lemma 6.8) are analytic on $\hat{W}$ so is the quotient $I_n\tau_n/\gamma_n^2$ away from the set of potentials where γ_n vanishes. Next we analyze I_n on a neighborhood of a potential with $\gamma_n = 0$. Note that by (8.6), (8.3) and since by Lemma 8.2, $\int_{\Gamma_n}\frac{\dot{\Delta}(\lambda,v)}{\sqrt[c]{\chi_p(\lambda,v)}}\,\mathrm{d}\lambda = 0$, one has

$$I_n(v) = \frac{4}{\pi}\int_{\Gamma_n}\big(\log(\lambda) - \log(\dot{\lambda}_n)\big)\dot{F}(\lambda,v)\,\mathrm{d}\lambda.$$

By Lemma 6.11 and (6.50) - (6.51), (6.55) -(6.56) one has

$$\begin{aligned}\dot{F}(\lambda,v) =& \frac{\dot{\Delta}(\lambda,v)}{\sqrt[c]{\chi_p(\lambda,v)}}\\ =& -\mathrm{i}\frac{\sqrt[c]{\chi_1(0,v)}\cdot\left(1-\left(\frac{\dot{\lambda}_*}{\lambda}\right)^2\right)c_{\dot{\Delta}}(v)\dot{\Delta}_2(\lambda,v)}{\sqrt[c]{\chi_1\left(-\frac{1}{16\lambda},-q,p\right)}}\cdot\prod_{m\in\mathbb{Z}}\frac{\dot{\lambda}_{1,m}-\lambda}{w_{1,m}(\lambda,v)}.\end{aligned}$$

For $n \geq 0$ define

$$\zeta_n(\lambda,v) := \prod_{m\neq n}\frac{\dot{\lambda}_{1,m}-\lambda}{w_{1,m}(\lambda,v)}$$

and

$$R(\lambda,v) := \frac{\sqrt[c]{\chi_1(0,v)}\left(1-\left(\frac{\dot{\lambda}_*}{\lambda}\right)^2\right)c_{\dot{\Delta}}(v)\dot{\Delta}_2(\lambda,v)}{\sqrt[c]{\chi_1\left(-\frac{1}{16\lambda},-q,p\right)}}, \tag{8.8}$$

Then $\dot{F}(\lambda,v) = -\mathrm{i}\frac{\dot{\lambda}_n-\lambda}{w_n(\lambda,v)}\zeta_n(\lambda,v)R(\lambda,v)$ where for brevity we write for $n \geq 0$, $\dot{\lambda}_n = \dot{\lambda}_{1,n}$ and $w_n(\lambda,v) := w_{1,n}(\lambda,v)$. Note that for any $v \in H_r^1$, ζ_n and R are analytic on $U_n \times V_v$ where V_v is given by (6.4) (cf Lemma 6.18, Lemma 2.14, Lemma 6.11). Hence

$$\begin{aligned}I_n(v) =& \frac{4}{\pi}\int_{\Gamma_n}\Big(\log(\lambda)-\log(\dot{\lambda}_n)\Big)\dot{F}(\lambda,v)\,\mathrm{d}\lambda = \frac{4}{\pi}\int_{\Gamma_n}\log\left(1+\frac{\lambda-\dot{\lambda}_n}{\dot{\lambda}_n}\right)\dot{F}(\lambda,v)\,\mathrm{d}\lambda\\ =& -\frac{4\mathrm{i}}{\pi}\int_{\Gamma_n}\log\left(1-\frac{\dot{\lambda}_n-\lambda}{\dot{\lambda}_n}\right)\frac{\dot{\lambda}_n-\lambda}{w_n(\lambda,v)}R(\lambda,v)\zeta_n(\lambda,v)\,\mathrm{d}\lambda.\end{aligned}$$

Hence collapsing Γ_n to G_n one obtains

$$I_n = -\frac{8\mathrm{i}}{\pi}\int_{\lambda_n^-}^{\lambda_n^+}\log\left(1-\frac{\dot{\lambda}_n-\lambda}{\dot{\lambda}_n}\right)\frac{\dot{\lambda}_n-\lambda}{w_n(\lambda,v)}R(\lambda,v)\zeta_n(\lambda,v)\,\mathrm{d}\lambda$$

where w_n is evaluated on the lower side of G_n, i.e., on $\lambda_n^- + t(\lambda_n^+ - \lambda_n^-)(1-\mathrm{i}0)$, $0 \leq t \leq 1$. The substitution

$$\lambda = \tau_n + t\gamma_n/2, \; -1 \leq t \leq 1, \qquad t_n = \frac{\dot{\lambda}_n - \tau_n}{\gamma_n/2}$$

then leads to $w_n(\lambda) = \mathrm{i}\frac{\gamma_n}{2}\sqrt[+]{1-t^2}$, $\dot{\lambda}_n - \lambda = (t_n - t)\gamma_n/2$ and thus

$$I_n = -\frac{8\mathrm{i}}{\pi}\int_{-1}^{1}\log\left(1-\frac{(t_n-t)\gamma_n}{2\dot{\lambda}_n}\right)\frac{t_n-t}{\mathrm{i}\sqrt[+]{1-t^2}}R(\lambda,v)\zeta_n(\lambda,v)\frac{\gamma_n}{2}\,\mathrm{d}t.$$

Since
$$-\frac{t_n - t}{\gamma_n}\log\left(1 - \frac{(t_n - t)\gamma_n}{2\dot\lambda_n}\right) = \frac{(t_n - t)^2}{2\dot\lambda_n} + O(\frac{1}{\dot\lambda_n^2}(t_n - t)^3\gamma_n)$$
it follows that
$$\frac{I_n\tau_n}{\gamma_n^2} = \frac{2\tau_n}{\pi\dot\lambda_n}\int_{-1}^{1}\frac{(t_n - t)^2(1 + o(\gamma_n))}{\sqrt[+]{1 - t^2}}R(\lambda)\zeta_n(\lambda)\,\mathrm{d}t. \tag{8.9}$$
Recall that by Lemma 6.15, $\dot\lambda_n = \tau_n + \gamma_n^2\ell_n^2$. It implies that $t_n \to 0$ as $\gamma_n \to 0$. Hence
$$\lim_{\gamma_n\to 0}\frac{I_n\tau_n}{\gamma_n^2} = \frac{2\tau_n}{\pi}\int_{-1}^{1}\frac{t^2}{\dot\lambda_n\sqrt[+]{1 - t^2}}R(\tau_n)\zeta_n(\tau_n)\,\mathrm{d}t = R(\tau_n)\zeta_n(\tau_n). \tag{8.10}$$
This shows that $I_n\tau_n/\gamma_n$ can be continuously extended to all of V_v. Since by Proposition 6.10, Z_n is a real analytic subvariety of $\hat W$, $I_n\tau_n/\gamma_n$ is continuous on $\hat W$ and analytic on $\hat W \setminus Z_n$ and on Z_n by the considerations above. It then follows from Theorem A.6 that $I_n\tau_n/\gamma_n$ is analytic on $\hat W$.

To derive the claimed asymptotics note that
$$R(\infty, v) = \lim_{|\lambda|\to\infty} R(\lambda, v) = \frac{\sqrt{\chi_1(0,q,p)}}{\sqrt{\chi_1(0,-q,p)}}\lim_{\lambda\to\infty} c_{\dot\Delta}(v)\dot\Delta_2(\lambda, v)$$
where by (6.27), $\lim_{\lambda\to\infty} c_{\dot\Delta}(v)\dot\Delta_2(\lambda, v) = 1$ and by Lemma 6.14, $\sqrt{\chi_1(0,v)} = \sqrt{\chi_1(0,-q,p)}$. Hence $R(\infty, v) = 1$ and $\mu \mapsto R(1/\mu, v)$ is analytic near $\mu = 0$ meaning that $R(\lambda, v) = 1 + O(1/\lambda)$ as $\lambda \to \infty$. Hence uniformly for $\lambda \in U_n$, $R(\lambda, v) = 1 + \ell_n^2$ as $n \to \infty$. Furthermore by Corollary 6.24, one has $\zeta_n(\lambda) = 1 + \ell_n^2$ as $n \to \infty$ uniformly for $\lambda \in U_n$ and by Lemma 3.17, $\gamma_n = \ell_n^2$. Hence by (8.9)
$$\frac{I_n\tau_n}{\gamma_n^2} = \frac{2\tau_n}{\pi\dot\lambda_n}\int_{-1}^{1}\frac{(t_n - t)^2(1 + \ell_n^2)}{\sqrt[+]{1 - t^2}}\,\mathrm{d}t$$

Finally by Lemma 6.15, $\frac{\tau_n}{\dot\lambda_n} = 1 + \ell_n^2$ and since $\int_{-1}^{1}\frac{t^2}{\sqrt[+]{1-t^2}}\,\mathrm{d}t = \frac{\pi}{2}$, $t_n = o(\gamma_n)$, $\gamma_n = \ell_n^2$, the claimed asymptotics follow.

To show that on $\hat W$, the real part of $I_n\tau_n/\gamma_n^2$ is positive for any $n \geq 0$ we first consider potentials in H_r^1. Recall that by Proposition 8.6(iii), I_n is real on H_r^1 and if $\gamma_n \neq 0$ positive. Furthermore if $\gamma_n = 0$, identity (8.9) holds, $I_n\tau_n/\gamma_n^2 = R(\tau_n, v)\zeta_n(\tau_n, v)$. By Definition 6.16, for any $m \geq 0$ with $m \neq n$, $(\dot\lambda_{1,m} - \lambda)/w_{1,m}(\lambda)$ is positive on G_n, implying that $\zeta_n(\tau_n) > 0$. Concerning $R(\tau_n, v)$, given in (8.8), first note that by Lemma 6.2, $\dot\lambda_* \in \mathrm{i}\mathbb{R}_{>0}$ and hence $1 - \left(\frac{\dot\lambda_*}{\tau_n}\right)^2 = 1 + \frac{(\dot\lambda_*)^2}{\tau_n^2} \geq 1$. By Lemma 6.20(i), $\sqrt[c]{\chi_1(0)} > 0$ and since $\tau_n > \lambda_{-1}^+$, (6.39) yields $\sqrt[c]{\chi_1(-\frac{1}{16\,\tau_n}, -q, p)} > 0$. Finally by the definition of $c_{\dot\Delta}\dot\Delta_2$ (cf Lemma 6.11), $c_{\dot\Delta}\dot\Delta_2(\tau_n) > 0$. This shows that $R(\tau_n, v) > 0$ as well. Altogether we thus proved that $I_n\tau_n/\gamma_n^2 > 0$ on H_r^1. It implies that any potential in H_r^1 admits a neighborhood in H_c^1 on which $\mathrm{Re}(I_n\tau_n/\gamma_n^2) > 0$. Moreover by (8.7), $I_n\tau_n/\gamma_n^2 = 1 + \ell_n^2$ for $n \to \infty$. By shrinking $\hat W$ if needed, it then follows that $\mathrm{Re}(I_n\tau_n/\gamma_n^2) > 0$ on $\hat W$. Hence the square root $\xi_n = \sqrt[+]{I_n\tau_n/\gamma_n^2}$ is well defined and real analytic on $\hat W$.

To compute $I_n\tau_n/\gamma_n$ at $v = 0$ note that $\tau_n = \dot\lambda_n$, $\gamma_n = 0$, $w_n(\lambda) = \tau_n - \lambda$, and by Lemma 8.1(ii)
$$R(\lambda)\zeta_n(\lambda) = \mathrm{i}\dot F(\lambda, 0)\frac{w_n(\lambda)}{\dot\lambda_n - \lambda} = \dot\omega(\lambda) = 1 + \frac{1}{16\,\lambda^2}.$$
By Lemma 3.10, τ_n and $1/16\tau_n$ at $v = 0$ are given by
$$\tau_n = \frac{n\pi}{2} + \frac{1}{4}\sqrt[+]{4n^2\pi^2 + 1}, \quad \frac{1}{16\tau_n} = -\frac{n\pi}{2} + \frac{1}{4}\sqrt[+]{4n^2\pi^2 + 1}$$
and
$$1 + \frac{1}{16\tau_n^2} = \frac{1}{2\,\tau_n}\sqrt[+]{4n^2\pi^2 + 1}.$$
Therefore at $v = 0$, ξ_n is given by
$$\xi_n(0) = \sqrt{I_n\tau_n/\gamma_n^2} = \sqrt[+]{\frac{1}{2\tau_n}\sqrt[+]{4n^2\pi^2 + 1}}.$$

(ii) Going through the arguments of the proof of the asymptotics for $I_n\tau_n/\gamma_n^2$ in item(i) one sees that the claimed asymptotics for $J_{1,n}$ hold. □

The next result concerns trace formulas which express the Hamiltonians H_m, $m \geq 1$, of the Sinh-Gordon hierarchy (2.46) - (2.49) in terms of the action variables $J_{k,n}$. First we need to establish the following auxiliary results.

***Lemma* 8.9** *(i) The sum $\sum_{n\in\mathbb{Z}} I_n$ converges locally uniformly on $\hat{W}$ and hence it is real analytic.*

(ii) For $k \geq 1$, $\sum_{n\geq 0} J_{k,n}$ and $\sum_{n\geq 0} J_{-k,-n}$ converge locally uniformly on $\hat{W} \cap H_c^{(k+1)/2}$ and hence are analytic.

(iii) For $k \geq 1$, $\sum_{n\geq 0} J_{k,-n}$ and $\sum_{n\geq 0} J_{-k,n}$ converge locally uniformly on $\hat{W}$ and hence are analytic.

Proof. (i) By Theorem 8.8(i), I_n satisfies

$$I_n = \frac{\gamma_n^2}{\tau_n}(1+\ell_n^2)$$

for $n \to \infty$ on $\hat{W}$. Furthermore, by Proposition 8.6(ii), $I_{-n}(v) = I_n(-q,p)$ on $\hat{W}$. Since by Lemma 3.17, $\tau_n = n\pi + \ell_n^2$ and $\gamma_n = \ell_n^2$ locally uniformly in $\hat{W}$ as $n \to \infty$, it then follows that $\sum_n I_n$ converges locally uniformly on $\hat{W}$ and hence is analytic (cf [6], Theorem A.4).

(ii) Let $k \geq 2$ be given. Expanding λ^k at $\dot{\lambda}_n$ by writing $\lambda = \dot{\lambda}_n + (\lambda - \dot{\lambda}_n)$ one has

$$\lambda^k - (\dot{\lambda}_n)^k = k(\dot{\lambda}_n)^{k-1}(\lambda - \dot{\lambda}_n) + \binom{k}{2}(\dot{\lambda}_n)^{k-2}(\lambda - \dot{\lambda}_n)^2 + \cdots .$$

Substituting this expression into the identity (8.5) for $J_{k,n}(v)$ one gets

$$J_{k,n} = k4^k(\dot{\lambda}_n)^{k-1}J_{1,n} + \cdots .$$

Using the fact that for $n \to \infty$, $\dot{\lambda}_n = n\pi + \ell_n^2$ (Lemma 3.15), $J_{1,n} = O(\gamma_n^2)$ (Theorem 8.8(ii)) and arguing as in the proof of Theorem 8.8(i) one sees that each term in the latter expansion is (at most) of the order of $O(n^{k-1}\gamma_n^2)$. Since by Theorem 4.10, $\gamma_n\langle n\rangle^{\frac{k-1}{2}} = \ell_n^2$ locally uniformly on $H_c^{\frac{k+1}{2}}$ it follows that as $n \to \infty$, $\gamma_n^2\langle n\rangle^{k-1} = \ell_n^1$ locally uniformly on $H_c^{\frac{k+1}{2}}$ for any $k \geq 1$. Hence we have shown for any $k \geq 1$, $\sum_{n\geq 0} J_{k,n}$ is locally uniformly summable on $\hat{W} \cap H_c^{(k+1)/2}$ and hence is analytic there. The corresponding statement for $\sum_{n\geq 0} J_{-k,-n}$ then follows from Proposition 8.6(ii).

(iii) For $k \geq 1$, $n \geq 0$

$$k(4\dot{\lambda}_n)^k J_{-k,n}(v) = \frac{1}{\pi}\int_{\Gamma_n}\left(\frac{\dot{\lambda}_n}{\lambda}\right)^k \frac{\dot{\Delta}(\lambda,v)}{\sqrt[c]{\chi_p(\lambda,v)}}\,\mathrm{d}\lambda.$$

Since $\dot{\lambda}_n = n\pi + \ell_n^2$ one has for $\lambda \in G_n$, $(\dot{\lambda}_n/\lambda)^k = 1 + \ell_n^2$. One then aruges as in the proof of Theorem 8.8(i) to conclude that $(4\dot{\lambda}_n)^k J_{-k,n} = \gamma_n(1+\ell_n^2)$, implying that $\sum_{n\geq 0} J_{-k,n}$ converges locally uniformly on $\hat{W}$. The corresponding statement for $\sum_{n\geq 0} J_{k,-n}$ then follows from Proposition 8.6(ii). □

For any $r > 0$, denote by $B(r)$ the closed disc in $\mathbb{C}$ of radius r centered at 0,

$$B(r) = \{\, \lambda \in \mathbb{C} : \ |\lambda| \leq r \,\}.$$

The boundary $\partial B(r)$ is assumed to be counterclockwise oriented. For any $v \in \hat{W}$, choose $r = r(v)$ so that $|\lambda_n^{\pm}(v)| < r, \ \forall\, n \leq 0$, and $|\lambda_n^{\pm}(v)| > r, \ \forall\, n > 0$.

***Lemma* 8.10** *(i) For any $v \in \hat{W}$*

$$\sum_{n\geq 1} J_{1,n}(v) = \frac{1}{2\pi}\int_{\partial B(r)} 4F(\lambda,v)\,\mathrm{d}\lambda - 2H_1(v)$$

and

$$\sum_{n\geq 1} J_{-1,n}(v) = \frac{1}{2\pi}\int_{\partial B(r)} \frac{1}{\lambda}(4\lambda)^{-1}F(\lambda,v)\,\mathrm{d}\lambda - \frac{1}{4}.$$

(ii) For any $m \geq 1$ and $v \in \hat{W} \cap H_r^{m+1}$

$$\sum_{n\geq 1} J_{2m+1,n}(v) = \frac{1}{2\pi} \int_{\partial B(r)} \lambda^{-1}(4\lambda)^{2m+1} F(\lambda, v)\, \mathrm{d}\lambda + \frac{1}{2}(-4)^{m+1} H_{2m+1}(v)$$

and for any $m \geq 1$ and $v \in \hat{W}$

$$\sum_{n\geq 1} J_{-(2m+1),n}(v) = \frac{1}{2\pi} \int_{\partial B(r)} \lambda^{-1}(4\lambda)^{-(2m+1)} F(\lambda, v)\, \mathrm{d}\lambda.$$

Proof. (i) By Lemma 2.19 and Lemma 8.9 both sides of both identities are analytic on $\hat{W}$. Hence it is enough to prove the claimed identities for $v \in \hat{W} \cap H_c^s$ with s big enough such that by Theorem 2.20 (for $m = 0$),

$$\cosh(\sigma_1(\lambda, v)) = \Delta(\lambda) + O(\lambda^{-3}).$$

Since by Theorem 4.15, the set of right sided finite gap potentials in $\hat{W} \cap H_c^s$ is dense in $\hat{W} \cap H_c^s$, w.l.o.g. we assume that $v \in \hat{W} \cap H_c^s$ is a right sided finite gap potential. It means that $\gamma_n(v) = 0$ for n sufficiently large. By Corollary 8.3(i), in such a case, there exists $R > 0$ so that $F(\cdot, v)$ is analytic on $\mathbb{C}^* \setminus B(R)$. Since F is odd in λ and the substitution $\lambda \mapsto \nu = -\lambda$, mapping Γ_n to ${\Gamma_n}^-$ which is a counterclockwise oriented contour around $\{-\lambda_n^+, -\lambda_n^-\}$ and no other periodic eigenvalues, one has for any $k \in \mathbb{Z}$

$$(-1)^k \frac{1}{\pi} \int_{{\Gamma_n}^-} \mu^{-1}(4\mu)^k F(\mu, v)\, \mathrm{d}\mu = -\frac{1}{\pi} \int_{\Gamma_n} \lambda^{-1}(4\lambda)^k F(\lambda, v)\, \mathrm{d}\lambda = J_{k,n}(v). \tag{8.11}$$

Since by Proposition 8.6(iii), $J_{1,n}(v) = 0$ whenever $\gamma_n = 0$, the sum $\sum_{n\geq 1} J_{1,n}$ is finite and by Cauchy's theorem, (8.2) and (8.11)

$$\begin{aligned}\sum_{n\geq 1} J_{1,n}(v) &= -\sum_{n\geq 1} \frac{1}{\pi} \int_{\Gamma_n} \frac{1}{\lambda}(4\lambda) F(\lambda, v)\, \mathrm{d}\lambda \\ &= -\frac{1}{2\pi} \int_{\partial B(R)} 4F(\lambda, v)\, \mathrm{d}\lambda + \frac{1}{2\pi} \int_{\partial B(r)} 4F(\lambda, v)\, \mathrm{d}\lambda.\end{aligned}$$

To compute $\mathrm{i}\,\mathrm{Res}_{\lambda=\infty}\, 4F(\lambda, v) = -\frac{1}{2\pi} \int_{\partial B(R)} 4F(\lambda, v)\, \mathrm{d}\lambda$, recall that by Proposition 8.5(i) $\cosh(F(\lambda, v)) = \Delta(\lambda, v)$, and that by Theorem 2.20,

$$\cosh(\sigma_1(\lambda, v)) = \Delta(\lambda) + O(\lambda^{-3})$$

where $\sigma_1(\lambda, v) = -\mathrm{i}\lambda + \frac{H_1(v)}{2\mathrm{i}\lambda}$ and $\lambda \in \Lambda_\tau = \{ \lambda \in \mathbb{C}^* : |\mathrm{Im}\lambda| \leq \tau,\ |\lambda| \geq 1 \}$. It then follows that F has a Laurent expansion of the form $F(\lambda, v) = \pm\sigma_1(\lambda, v) + O(\lambda^{-3})$. Since by Corollary 8.3(ii) $F(\lambda_n^\pm, v) = -\mathrm{i}n\pi$ for any $n \in \mathbb{Z}$, we see that the Laurent expansion at ∞ of F is of the form $F(\lambda, v) = \sigma_1(\lambda, v) + O(\lambda^{-3})$. It implies that

$$\mathrm{i}\,\mathrm{Res}_{\lambda=\infty}\, 4F(\lambda, v) = -\mathrm{i}\,\mathrm{Res}_0\, \frac{4}{z^2} F(\frac{1}{z}, v) = -2H_1(v)$$

and in turn

$$\sum_{n\geq 1} J_{1,n}(v) = -2H_1(v) + \frac{1}{2\pi} \int_{\partial B(r)} 4F(\lambda, v)\, \mathrm{d}\lambda.$$

Since $\mathrm{i}\,\mathrm{Res}_{\lambda=\infty} \left(\frac{1}{\lambda}(4\lambda)^{-1} F(\lambda, v)\right) = -1/4$ one sees by using the same arguments that

$$\sum_{n\geq 1} J_{-1,n}(v) = -1/4 + \frac{1}{2\pi} \int_{\partial B(r)} \frac{1}{\lambda}(4\lambda)^{-1} F(\lambda, v)\, \mathrm{d}\lambda.$$

(ii) We argue similarly as in the proof of item (i) to obtain

$$\begin{aligned}\sum_{n\geq 1} J_{2m+1,n}(v) &= -\sum_{n\geq 1} \frac{1}{\pi} \int_{\Gamma_n} \frac{1}{\lambda}(4\lambda)^{2m+1} F(\lambda, v)\, \mathrm{d}\lambda \\ &= -\frac{1}{2\pi} \int_{\partial B(R)} \frac{1}{\lambda}(4\lambda)^{2m+1} F(\lambda, v)\, \mathrm{d}\lambda + \frac{1}{2\pi} \int_{\partial B(r)} \frac{1}{\lambda}(4\lambda)^{2m+1} F(\lambda, v)\, \mathrm{d}\lambda \\ &= \mathrm{i}\,\mathrm{Res}_{\lambda=\infty} \frac{1}{\lambda}(4\lambda)^{2m+1} F(\lambda, v) + \frac{1}{2\pi} \int_{\partial B(r)} \frac{1}{\lambda}(4\lambda)^{2m+1} F(\lambda, v)\, \mathrm{d}\lambda.\end{aligned}$$

Taking $v \in \hat{W} \cap H^s_c$ for s big enough one has by Theorem 2.20, $F(\lambda) = \sigma_{2m+1}(\lambda) + O(\lambda^{-(2m+3)})$ and

$$\sigma_{2m+1}(\lambda) = -\mathrm{i}\lambda - \mathrm{i}\sum_{n=0}^{m} \frac{(-1)^n H_{2n+1}(v)}{(2\lambda)^{2n+1}}$$

one sees that

$$\mathrm{i}\,\mathrm{Res}_{\lambda=\infty}\,\frac{1}{\lambda}(4\lambda)^{2m+1}F(\lambda,v) = -\mathrm{i}\,\mathrm{Res}_0\Big(\frac{1}{z}(4/z)^{2m+1}F\big(\frac{1}{z},v\big) = (-1)^{m+1}2\cdot 4^m H_{2m+1}(v).$$

and hence in turn

$$\sum_{n\geq 1} J_{2m+1,n}(v) = \frac{1}{2\pi}\int_{\partial B(r)} \lambda^{-1}(4\lambda)^{2m+1}F(\lambda,v)\,\mathrm{d}\lambda + \frac{1}{2}(-4)^{m+1}H_{2m+1}(v).$$

By Lemma 8.9(iii), the above arguments also show that for any $m \geq 1$ and $v \in \hat{W}$

$$\sum_{n\geq 1} J_{-(2m+1),n}(v) = \frac{1}{2\pi}\int_{\partial B(r)} \lambda^{-1}(4\lambda)^{-(2m+1)}F(\lambda,v)\,\mathrm{d}\lambda.$$

□

***Theorem* 8.11** *(i) For $v = (q,p) \in \hat{W}$*

$$\sum_{n\in\mathbb{Z}} J_{1,n}(v) = -1/4 - 2H_1(v), \quad \sum_{n\in\mathbb{Z}} J_{-1,n}(q,p) = -1/4 - 2H_1(-q,p).$$

(ii) For any $m \geq 1$ and $v = (q,p) \in \hat{W} \cap H^{m+1}_c$

$$\sum_{n\in\mathbb{Z}} J_{2m+1,n}(v) = \frac{1}{2}(-4)^{m+1}H_{2m+1}(v), \quad \sum_{n\in\mathbb{Z}} J_{-(2m+1),n}(-q,p) = \frac{1}{2}(-4)^{m+1}H_{2m+1}(-q,p).$$

Proof. (i) For any given potential $v = (q,p) \in \hat{W}$, one has by Proposition 8.6(ii)

$$\sum_{n\leq -1} J_{1,n}(q,p) = \sum_{n\geq 1} J_{-1,n}(-q,p)$$

and by Lemma 8.10(i)

$$\sum_{n\geq 1} J_{-1,n}(-q,p) = \frac{1}{2\pi}\int_{\partial B(r(-q,p))} \frac{1}{\lambda}(4\lambda)^{-1}F(\lambda,-q,p)\,\mathrm{d}\lambda - \frac{1}{4}.$$

where $r \equiv r(-q,p) > 0$ is chosen so that $|\lambda^{\pm}_n(-q,p)| < r,\ \forall\, n \leq 0$, and $|\lambda^{\pm}_n(-q,p)| > r,\ \forall\, n > 0$. Using Lemma 8.10 once more to rewrite $\sum_{n\geq 1} J_{1,n}(q,p)$, then leads to the following identity

$$\sum_{n\in\mathbb{Z}} J_{1,n}(q,p) = -\frac{1}{4} - 2H_1(v) + J_{1,0}(q,p) + \frac{1}{2\pi}\int_{\partial B(r)} 4F(\lambda,v)\,\mathrm{d}\lambda + \frac{1}{2\pi}\int_{\partial B(r(-q,p))} \frac{1}{\lambda}(4\lambda)^{-1}F(\lambda,-q,p)\,\mathrm{d}\lambda.$$

We claim that

$$J_{1,0}(v) + \frac{1}{2\pi}\int_{\partial B(r)} 4F(\lambda,v)\,\mathrm{d}\lambda + \frac{1}{2\pi}\int_{\partial B(r(-q,p))} \frac{1}{\lambda}(4\lambda)^{-1}F(\lambda,-q,p)\,\mathrm{d}\lambda = 0. \tag{8.12}$$

To see it we make the substitution $\mu := \frac{1}{16\,\lambda}$ and recall that by Corollary 8.3(iii), $F(\lambda,-q,p) = -F(\mu,q,p)$. Furthermore, the circle $\partial B(r(-q,p))$ gets mapped to the circle $\partial^- B(1/16r(-q,p))$ where the minus sign in ∂^- indicates that the circle is clockwise oriented. Since $\frac{1}{\lambda}(4\lambda)^{-1} = 16\mu\cdot 4\mu$ and $\mathrm{d}\lambda = -\frac{1}{16\,\mu^2}\,\mathrm{d}\mu$ it follows that $\frac{1}{\lambda}(4\lambda)^{-1}\,\mathrm{d}\lambda = -\frac{1}{\mu}(4\mu)\,\mathrm{d}\mu$, yielding altogether

$$\frac{1}{2\pi}\int_{\partial B(r(-q,p))} \frac{1}{\lambda}(4\lambda)^{-1}F(\lambda,-q,p)\,\mathrm{d}\lambda = \frac{1}{2\pi}\int_{\partial^- B(1/16r(-q,p))} \frac{1}{\mu}(4\mu)F(\mu,q,p)\,\mathrm{d}\mu.$$

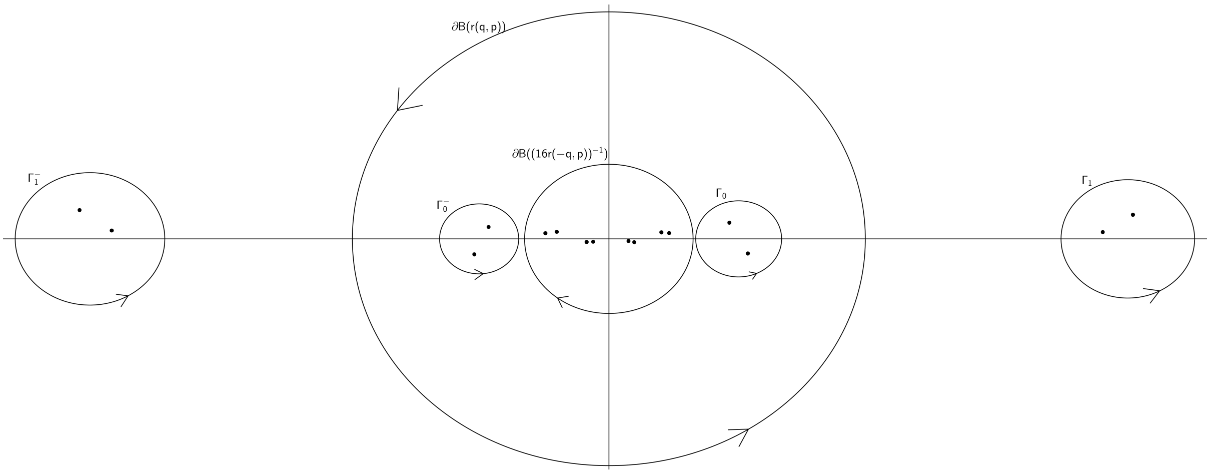

Figure 5: Illustration of Γ_0, ${\Gamma_0}^-$, $\partial B(r(q,p))$ and $\partial^- B((16r(-q,p))^{-1})$.

Since by the definition of $r(-q,p)$ and Lemma 6.6(ii), it then follows that $|\lambda_n^\pm(q,p)| > 1/16r(-q,p)$ for all $n \geq 0$, $|\lambda_n^\pm(q,p)| < r(-q,p)$ for all $n \leq -1$ as depicted in Figure 5. Hence by the analytic properties of F it then follows from Cauchy's theorem that

$$\frac{1}{2\pi}\int_{\partial B(r)} 4F(\lambda,v)\,\mathrm{d}\lambda + \frac{1}{2\pi}\int_{\partial^- B(1/16r(-q,p))} \frac{1}{\lambda}(4\lambda)F(\lambda,q,p)\,\mathrm{d}\lambda = \frac{1}{2\pi}\int_{\Gamma_0\cup{\Gamma_0}^-} 4F(\lambda,v)\,\mathrm{d}\lambda.$$

Finally note that by (8.11) and (8.2)

$$\frac{1}{2\pi}\int_{\Gamma_0\cup{\Gamma_0}^-} 4F(\lambda,v)\,\mathrm{d}\lambda = -J_{1,0}$$

and (8.12) follows yielding

$$\sum_{n\in\mathbb{Z}} J_{1,n}(q,p) = -\frac{1}{4} - 2H_1(v).$$

By Proposition 8.6(ii), $J_{-k,n}(q,p) = J_{k,-n}(-q,p)$, implying that

$$\sum_{n\in\mathbb{Z}} J_{-1,n}(q,p) = -\frac{1}{4} - 2H_1(-q,p).$$

(ii) Using the same arguments as in the proof of item (i) one sees that the claimed identities hold. □

8.2 Psi-Functions

A key ingredient in the definition of our candidates for the canonical angle variables is a certain family of one forms on the spectral curve. The aim of this section is to construct this family. We begin with some preparations. Recall that according to Lemma B.1, for any given sequences $(\sigma_{1,k})_k$, $(\sigma_{2,k})_k$ in the space $\ell^* = \{\ \lambda_k = k\pi + \ell_k^2\ :\ \lambda_k \in \mathbb{C}^*,\ k\in\mathbb{Z}\ \}$ the infinite products

$$f_1(\lambda) := \prod_{k\in\mathbb{Z}} \frac{\sigma_{1,k}-\lambda}{\pi_k}, \quad f_2(\lambda) := \prod_{k\in\mathbb{Z}} \frac{\sigma_{2,k}+\frac{1}{16\lambda}}{\pi_k}$$

define analytic functions on $\mathbb{C}^*$ with roots $\sigma_{1,k}$ ($k\in\mathbb{Z}$), and respectively, $(-16\sigma_{2,k})^{-1}$ ($k\in\mathbb{Z}$). Actually, f_1 is analytic in $\lambda = 0$ and f_2 is analytic in $\lambda=\infty$. We denote by $f_2(\infty)$ the value of f_2 at ∞ and note that $f_2(\infty) = \prod_{k\in\mathbb{Z}} \frac{\sigma_{2,k}}{\pi_k} \in \mathbb{C}$. For any given $v_0 \in \hat{W}$, let $V_{v_0} \subset \hat{W}$ be a neighborhood of v_0, $(U_m)_{m\in\mathbb{Z}}$, U_* be a family of isolating neighborhoods and $(\Gamma_m)_{m\in\mathbb{Z}}$ be contours as in Section 6.2, which work for any $v\in V_{v_0}$. Then define

$$U_{1,m} := U_m\ (m\geq 0), \quad U_{1,-m} := -U_m\ (m\geq 1) \tag{8.13}$$

$$U_{2,m} := -U_{-m}\ (m \geq 0), \quad U_{2,-m} := U_{-m}\ (m \geq 1) \tag{8.14}$$

$$\Gamma_{1,m} := \Gamma_m\ (m \geq 0), \quad \Gamma_{1,-m} := \Gamma_m{}^-\ (m \geq 1) \tag{8.15}$$

$$\Gamma_{2,m} := \Gamma_{-m}{}^-\ (m \geq 0), \quad \Gamma_{2,-m} := \Gamma_{-m}\ (m \geq 1). \tag{8.16}$$

We further recall that

$$G_{1,m} = [\lambda_m^-, \lambda_m^+]\ (m \geq 0), \quad G_{1,-m} = -G_{1,m}\ (m \geq 1) \tag{8.17}$$

and

$$G_{2,m} = [-\lambda_{-m}^+, -\lambda_{-m}^-]\ (\mathfrak{m} \geq 0), \quad G_{2,-m} = -G_{2,m}\ (m \geq 1). \tag{8.18}$$

By Lemma 6.21, for any potential in V_{v_0}, $\sqrt[c]{\chi_p(\lambda)}$ is a nonvanishing analytic function on $\mathbb{C} \setminus \bigcup_{m\in\mathbb{Z}}(G_m \cup -G_m)$. Furthermore, one can choose contours $\Gamma_{j,m}$ inside $U_{j,m}$ and annular neighborhoods $U'_{j,m}$ of $\Gamma_{j,m}$, such that for any potential in V_{v_0}, maybe after shrinking V_{v_0} if necessary, $\overline{U'_{j,m}} \subset U_{j,m} \setminus (G_{j,m} \cup \{\mu_{j.m}\})$ and

$$\inf\{\ |\chi_p(\lambda)| :\ \lambda \in \bigcup_{j=1,2,\ m\in\mathbb{Z}} U'_{j,m}\ \} > 0, \quad \inf\{\ |\dot{m}_2(\lambda)| :\ \lambda \in \bigcup_{j=1,2,\ m\in\mathbb{Z}} U'_{j,m}\ \} > 0. \tag{8.19}$$

We consider analytic functions $\psi_n(\lambda)$, $n \geq 0$, on $\mathbb{C}^*$ which are given by the following infinite products

$$\psi_n(\lambda) := -\frac{1}{\pi_n} C_n \psi_{n,1}(\lambda) \cdot \psi_{n,2}(\lambda) \tag{8.20}$$

where

$$C_n \equiv C_{\psi_{n,2}} := 1/\psi_{n,2}(\infty) \tag{8.21}$$

$$\psi_{n,1}(\lambda) = \prod_{\substack{k\in\mathbb{Z}\\ k\neq n}} \frac{\sigma_{1,k} - \lambda}{\pi_k}, \quad \psi_{n,2}(\lambda) = \prod_{k\in\mathbb{Z}} \frac{\sigma_{2,k} + \frac{1}{16\lambda}}{\pi_k} \tag{8.22}$$

and $\sigma_{k,j} = k\pi + \ell_k^2$ satisfy

$$\sigma_{1,k} \in U_{1,k}, \quad (-16\sigma_{2,k})^{-1} \in U_{2,k}. \tag{8.23}$$

Note that under these assumptions the contour integrals

$$\int_{\Gamma_{j,m}} \frac{\psi_n(\lambda)}{\sqrt[c]{\chi_p(\lambda)}} \,\mathrm{d}\lambda \quad m \in \mathbb{Z},\ j = 1, 2$$

are well defined. In particular, they do not depend on the choice of the contours $\Gamma_{j,m}$ as long as $\Gamma_{j,m} \subset U_{j,m}$ and $G_{j,m}$ is inside the contour $\Gamma_{j,m}$. We want to determine $\sigma_{1,k}$, $\sigma_{2,k}$ in such a way that all these contour integrals vanish except the one for $\Gamma_{1,n}$.

***Theorem* 8.12** *For each potential v in some complex neighborhood $W \subset \hat{W}$ of H_r^1 with $\mathcal{S}_{rec}(W) = W$ and each $n \geq 0$, there exist analytic functions $\psi_n(\lambda) \equiv \psi_n(\lambda, v)$ of the form* (8.20)-(8.23) *so that for any $m \in \mathbb{Z}$,*

$$\frac{1}{2\pi} \int_{\Gamma_{1,m}} \frac{\psi_n(\lambda)}{\sqrt[c]{\chi_p(\lambda)}} \,\mathrm{d}\lambda = \delta_{nm}, \tag{8.24}$$

$$\frac{1}{2\pi} \int_{\Gamma_{2,m}} \frac{\psi_n(\lambda)}{\sqrt[c]{\chi_p(\lambda)}} \,\mathrm{d}\lambda = 0. \tag{8.25}$$

The possibly complex roots $\sigma_{1,k}^n$ $(k \neq n)$ and $(-16\sigma_{2,k}^n)^{-1}$ $(k \in \mathbb{Z})$ of $\psi_n(\lambda)$ depend analytically on v in such a way that

$$\sigma_{j,k}^n - \tau_{j,k} = \gamma_{j,k}^2 \ell_k^2 \quad \forall\ k \in \mathbb{Z},\ j = 1, 2 \tag{8.26}$$

where

$$\tau_{j,k} := (\lambda_{j,k}^+ + \lambda_{j,k}^-)/2, \quad \sigma_{1,n}^n := \tau_n. \tag{8.27}$$

Furthermore, if v is real valued, then $\lambda_{j,k}^- \leq \sigma_{j,k}^n \leq \lambda_{j,k}^+$ for all $k \in \mathbb{Z}$, $j = 1, 2$.

Remark 8.13. One can show that the functions $\psi_n(\lambda)$ of the form (8.20)-(8.23) are uniquely determined by (8.24)-(8.25). For a precise statement see Proposition 8.23.

Remark 8.14. By the reciprocity property, Theorem 8.12 can be used to construct functions $\psi_n^{rec}(\lambda)$, $n \geq 0$, of the type (8.20)-(8.23) with the roles of $\psi_{n,1}$ and $\psi_{n,2}$ interchanged so that for any $m \in \mathbb{Z}$

$$\frac{1}{2\pi}\int_{\Gamma_{1,m}} \frac{\psi_n^{rec}(\lambda)}{\sqrt[c]{\chi_p(\lambda)}}\,\mathrm{d}\lambda = 0, \quad \frac{1}{2\pi}\int_{\Gamma_{2,m}} \frac{\psi_n^{rec}(\lambda)}{\sqrt[c]{\chi_p(\lambda)}}\,\mathrm{d}\lambda = \delta_{nm}.$$

Indeed, define for $n \geq 0$, and $v = (q,p) \in W$

$$\psi_n^{rec}(\lambda, v) := \frac{1}{16\lambda^2}\psi_n(-\frac{1}{16\lambda}, -q, p).$$

Since $\sqrt[c]{\chi_p(-\frac{1}{16\lambda}, q, p)} = \sqrt[c]{\chi_p(\lambda, -q, p)}$ (Lemma 6.21), the change of variable $\lambda := -\frac{1}{16\mu}$ in the following integrals yields for any $m \in \mathbb{Z}$

$$J_{1,n,m} := \int_{\Gamma_{1,m}(q,p)} \frac{\psi_n^{rec}(\lambda, q, p)}{\sqrt[c]{\chi_p(\lambda, q, p)}}\,\mathrm{d}\lambda = \int_{\Gamma_{2,m}(-q,p)} \frac{\psi_n(\mu, -q, p)}{\sqrt[c]{\chi_p(\mu, -q, p)}}\,\mathrm{d}\mu = 0$$

and

$$J_{2,n,m} := \int_{\Gamma_{2,m}(q,p)} \frac{\psi_n^{rec}(\lambda, q, p)}{\sqrt[c]{\chi_p(\lambda, q, p)}}\,\mathrm{d}\lambda = \int_{\Gamma_{1,m}(-q,p)} \frac{\psi_n(\mu, -q, p)}{\sqrt[c]{\chi_p(\mu, -q, p)}}\,\mathrm{d}\mu = 2\pi\delta_{nm}.$$

Remark 8.14 leads to the following application.

***Corollary* 8.15** *For any $n \geq 1$ and $v = (q,p) \in W$,*

$$\psi_{-n}(\lambda, q, p) := \psi_n(\frac{1}{16\lambda}, -q, p)\frac{1}{16\lambda^2}$$

satisfies for any $m \in \mathbb{Z}$

$$\int_{\Gamma_{1,m}} \frac{\psi_{-n}(\lambda, v)}{\sqrt[c]{\chi_p(\lambda, v)}}\,\mathrm{d}\lambda = 0, \quad \int_{\Gamma_{2,m}} \frac{\psi_{-n}(\lambda, v)}{\sqrt[c]{\chi_p(\lambda, v)}}\,\mathrm{d}\lambda = 2\pi\delta_{-n,m}.$$

The possibly complex roots $\sigma_{1,k}^{-n}$ ($k \in \mathbb{Z}$) and $(-16\sigma_{2,k}^{-n})^{-1}$ ($k \neq -n$) of $\psi_{-n}(\lambda)$ depend analytically on v and satisfy

$$\sigma_{j,k}^{-n}(v) - \tau_{j,k}(v) = \gamma_{j,k}(v)^2\ell_k^2 \quad \forall\ (j,k) \neq (2,-n).$$

Furthermore, if v is real valued, then

$$\lambda_{j,k}^-(v) \leq \sigma_{j,k}^{-n}(v) \leq \lambda_{j,k}^+(v) \quad \forall\ (j,k) \neq (2,-n).$$

Proof. Consider the case $j = 2$, $m \in \mathbb{Z}$. By the change of variable $\mu = \frac{1}{16\lambda}$

$$\int_{\Gamma_{2,m}(q,p)} \frac{\psi_{-n}(\lambda, v)}{\sqrt[c]{\chi_p(\lambda, v)}}\,\mathrm{d}\lambda = \int_{\Gamma_{2,m}(q,p)} \frac{\psi_n(\frac{1}{16\lambda}, -q, p)}{\sqrt[c]{\chi_p(\lambda, q, p)}}\frac{\mathrm{d}\lambda}{16\lambda^2} = \int_{\Gamma_{1,-m}(-q,p)} \frac{\psi_n(\mu, -q, p)}{\sqrt[c]{\chi_p(\frac{1}{16\mu}, q, p)}}(-1)\,\mathrm{d}\mu.$$

Since $\sqrt[c]{\chi_p(\frac{1}{16\mu}, q, p)} = -\sqrt[c]{\chi_p(\mu, -q, p)}$ and it then follows that

$$\int_{\Gamma_{2,-m}(q,p)} \frac{\psi_{-n}(\lambda, v)}{\sqrt[c]{\chi_p(\lambda, v)}}\,\mathrm{d}\lambda = \int_{\Gamma_{1,-m}(-q,p)} \frac{\psi_n(\mu, -q, p)}{\sqrt[c]{\chi_p(\mu, -q, p)}}\,\mathrm{d}\mu = 2\pi\delta_{-n,m}.$$

The second identity is of course verified in the same way. All other claims follow from Theorem 8.12. □

In the remainder of this section we prove Theorem 8.12 by adapting the strategy used to prove corresponding results for the KdV equation [8] and the defocusing NLS equation [6] to the more complicated situation at hand. First we reformulate the statement of Theorem 8.12 as a functional equation which then is solved by the means of the implicit function theorem. It turns out that the form of the product representation (8.20)-(8.23) of $\psi_n(\lambda)$ as well as the estimates for $\sigma_{j,k}^n$ are important features for the construction of the angles.

Using a different approach, the existence of such functions in the case where v is real valued was first studied in [17]. The construction of these functions on a complex neighborhood of H_r^1 is not straightforward since the zeroes $\sigma_{1,k}^n$, $(-16\sigma_{2,k}^n)^{-1}$ are not confined to the gaps anymore (cf also [11]).

To define the functional equation mentioned above, introduce

$$\ell^2_{\hat{n}} := \ell^2(\mathbb{Z} \setminus \{n\},\ \mathbb{C}).$$

and denote by Ω the open subset of $\hat{W} \times \ell^2 \times \ell^2$ of elements (v, s_1, s_2) so that

$$\sigma_{1,k} := k\pi + s_{1,k} \in U_{1,k} \quad \forall\ k \in \mathbb{Z}. \tag{8.28}$$

$$\sigma_{2,k} := k\pi + s_{2,k} \in \mathbb{C}^* \quad \text{with} \quad (-16\sigma_{2,k})^{-1} \in U_{2,k} \ \ \forall\ k \in \mathbb{Z}. \tag{8.29}$$

Here v is an element of $V_{v_0} \subset \hat{W}$ for some potential $v_0 \in \hat{W}$ and $U_{j,k}$ are the open subsets defined by (8.13)-(8.14) in terms of isolating neighborhoods U_m, $m \in \mathbb{Z}$, for the neighborhood $V_{v_0} \subset \hat{W}$ of $v_0 \in \hat{W}$. Since by Lemma 8.16 below, the functions ψ_n we are going to construct for $v \in H^1_r$ have to have roots in the gap intervals $G_{1,m}$ ($m \neq n$) and $G_{2,m}$ ($m \in \mathbb{Z}$), it suffices to consider Ω. In addition, define

$$\Omega_n := \{\ (v, s_1, s_2) \in \Omega :\ s_{1,n} = \tau_{1,n} - n\pi\ \}.$$

By a slight abuse of notation, for any $(v, s_1, s_2) \in \Omega_n$ we also denote $(s_{1,k})_{k\neq n}$ by s_1.

Given $v \in \hat{W}$ and $n \in \mathbb{Z}$, we are looking for a solution $s_1 = (s_{1,k})_{k\neq n} \in \ell^2_{\hat{n}}$ and $s_2 = (s_{2,k})_{k\in\mathbb{Z}} \in \ell^2$ with $(v, s_1, s_2) \in \Omega_n$ to the equation

$$F^n(v, s_1, s_2) = (F^n_1(v, s_1, s_2), F^n_2(v, s_1, s_2)) = 0 \tag{8.30}$$

where $F^n_1 = (F^n_{1,m})_{m\neq n}$ and $F^n_2 = (F^n_{2,m})_{m\in\mathbb{Z}}$ are defined as follows

$$F^n_{1,m} := A^n_{1,m}(v) f_n(s_1, s_2) := (n - m) \int_{\Gamma_{1,m}} \frac{f_n(s_1, s_2, \lambda)}{\sqrt[c]{\chi_p(\lambda)}}\, \mathrm{d}\lambda \tag{8.31}$$

$$F^n_{2,m} := A^n_{2,m}(v) f_n(s_1, s_2) := 16\pi^2_m \pi_n \int_{\Gamma_{2,m}} \frac{f_n(s_1, s_2, \lambda)}{\sqrt[c]{\chi_p(\lambda)}}\, \mathrm{d}\lambda. \tag{8.32}$$

Here

$$f_n(s_1, s_2, \lambda) := -\frac{1}{\pi_n} \frac{1}{f_{n,2}(s_2, \infty)} f_{n,1}(s_1, \lambda) f_{n,2}(s_2, \lambda),$$

$$f_{n,1}(s_1, \lambda) := \prod_{k\neq n} \frac{\sigma_{1,k} - \lambda}{\pi_k}, \quad f_{n,2}(s_2, \lambda) := \prod_{k\in\mathbb{Z}} \frac{\sigma_{2,k} + \frac{1}{16\lambda}}{\pi_k}. \tag{8.33}$$

and

$$\sigma_{1,k} := k\pi + s_{1,k}\ (k \neq n), \quad \sigma_{2,k} := k\pi + s_{2,k}\ (k \in \mathbb{Z}).$$

Note that since $(v, s_1, s_2) \in \Omega_n$, $f_{n,2}(s_2, \infty) = \prod_{k\in\mathbb{Z}} \frac{\sigma_{2,k}}{\pi_k} \in \mathbb{C}^*$ and hence $f_n(s_1, s_2, \lambda)$ is well defined.

By a slight abuse of notation, for any $n \geq 0$ and $m \in \mathbb{Z}$, we view $F^n_{1,m}$ ($m \neq n$) and $F^n_{2,m}$ either as a function on Ω_n or Ω.

The reason we make the ansatz (8.20) - (8.23) for $\psi_n(\lambda)$ stems from the following observation for real valued potentials.

***Lemma* 8.16** *Let $j \in \{1, 2\}$, $m \in \mathbb{Z}$ (with $m \neq n$ in the case $j = 1$), and $v \in H^1_r$. Furthermore assume that $f : U_{j,m} \to \mathbb{C}$ is real analytic. If $A_{j,m}(v)f = 0$, then f has a root in $G_{j,m}$.*

Proof. Let us first treat the case $j = 1$. If $G_{1,m}$ is not a single point, then we may shrink the contour $\Gamma_{1,m}$ to the interval $G_{1,m}$. By (6.34) (Lemma 6.20), one has $(-1)^{n+1} \sqrt[c]{\chi_p(\lambda)} > 0$ for $\lambda^-_{1,m} < \lambda < \lambda^+_{1,m}$. Hence

$$0 = \int_{\Gamma_{1,m}} \frac{f(\lambda)}{\sqrt[c]{\chi_p(\lambda)}}\, \mathrm{d}\lambda = 2(-1)^{n+1} \int_{\lambda^-_{1,m}}^{\lambda^+_{1,m}} \frac{f(\lambda)}{\sqrt[+]{\chi_p(\lambda)}}\, \mathrm{d}\lambda.$$

Since by assumption $f(\lambda) \in \mathbb{R}$ for $\lambda \in G_{1,m}$, the latter integral can only vanish if $f(\lambda) \equiv 0$ on $G_{1,m}$ (and hence on $U_{1,m}$) or f changes sign on the open interval $(\lambda^-_{1,m}, \lambda^+_{1,m})$ at least once. If on the other hand $G_{1,m}$ is a single point, $G_{1,m} = \{\tau_{1,m}\}$, then we may extract the factor $\tau_{1,m} - \lambda$ from $\sqrt[c]{\chi_p(\lambda)}$ to obtain a Cauchy integral around $\tau_{1,m}$ which gives $f(\tau_{1,m}) = 0$. The case $j = 2$ is treated in the same fashion, by using now (6.37) (Lemma 6.20). $\square$

In a first step, let us study the maps F^n in more detail.

***Lemma* 8.17** *For any $n \geq 0$, the formulas* (8.30) - (8.33) *define a real analytic map*

$$F^n : \Omega_n \to \ell^2_{\hat{n}} \times \ell^2.$$

The maps F^n are locally uniformly bounded. More precisely,

$$\sum_{m \neq n} |F^n_{1,m}(v, s_1, s_2)|^2 + \sum_{m \in \mathbb{Z}} |F^n_{2,m}(v, s_1, s_2)|^2 \leq C$$

where $C > 0$ can be chosen uniformly in $n \geq 0$ and locally uniformly on Ω_n.

Before proving Lemma 8.17, we make some preliminary considerations.

***Lemma* 8.18** *Let $m \in \mathbb{Z}$ and $v \in \hat{W}$.*

(i) If $f : U_{1,m} \to \mathbb{C}$ is analytic, then

$$\frac{1}{2\pi}\Big| \int_{\Gamma_{1,m}} \frac{f(\lambda)}{w_{1,m}(\lambda)} \, \mathrm{d}\lambda \Big| \leq \max_{\lambda \in G_{1,m}} |f(\lambda)|.$$

Moreover, if $v \in H^1_r$ and f is real analytic then there exists $\nu \in G_{1,m}$ so that

$$\frac{1}{2\pi \mathrm{i}} \int_{\Gamma_{1,m}} \frac{f(\lambda)}{w_{1,m}(\lambda)} \, \mathrm{d}\lambda = -f(\nu).$$

(ii) If $f : U_{2,m} \to \mathbb{C}$ is analytic, then

$$\frac{1}{2\pi}\Big| \int_{\Gamma_{2,m}} \frac{f(\lambda)}{w_{2,m}(\lambda)} \, \mathrm{d}\lambda \Big| \leq \max_{\lambda \in G_{2,m}} 16|\lambda|^2 |f(\lambda)|.$$

Moreover, if $v \in H^1_r$ and f is real analytic then there exists $\nu \in G_{2,m}$ so that

$$\frac{1}{2\pi \mathrm{i}} \int_{\Gamma_{2,m}} \frac{f(\lambda)}{w_{2,m}(\lambda)} \, \mathrm{d}\lambda = -16\nu^2 f(\nu).$$

Proof. Item (i) is proved in [[6], Lemma 14.3]. Considering (ii), set $\tilde{\Gamma}_{2,m} := \{ -\frac{1}{16\lambda} \, : \, \lambda \in \Gamma_{2,m} \}$ and $\tilde{G}_{2,m} := \{ -\frac{1}{16\lambda} \, : \, \lambda \in G_{2,m} \}$. With the change of variable $\lambda := -\frac{1}{16\mu}$ one obtains

$$\frac{1}{2\pi \mathrm{i}} \int_{\Gamma_{2,m}} \frac{f(\lambda)}{w_{2,m}(\lambda)} d\lambda = \frac{1}{2\pi \mathrm{i}} \int_{\tilde{\Gamma}_{2,m}} \frac{f(-\frac{1}{16\mu})}{\sqrt[s]{(\lambda^+_{2,m} - \mu)(\lambda^-_{2,m} - \mu)}} \frac{\mathrm{d}\mu}{16\mu^2}.$$

By replacing f by $f(-\frac{1}{16\mu})/16\mu^2$ in (i) one gets

$$\frac{1}{2\pi \mathrm{i}}\Big| \int_{\tilde{\Gamma}_{2,m}} \frac{f(-\frac{1}{16\mu})}{\sqrt[s]{(\lambda^+_{2,m} - \mu)(\lambda^-_{2,m} - \mu)}} \frac{\mathrm{d}\mu}{16\mu^2} \Big| \leq \max_{\mu \in \tilde{G}_{2,m}} \frac{|f(-\frac{1}{16\mu})|}{16|\mu|^2} = \max_{\lambda \in G_{2,m}} 16|\lambda|^2 |f(\lambda)|$$

and, in case $v \in H^1_r$,

$$\frac{1}{2\pi \mathrm{i}} \int_{\tilde{\Gamma}_{2,m}} \frac{f(-\frac{1}{16\mu})}{\sqrt[s]{(\lambda^+_{2,m} - \mu)(\lambda^-_{2,m} - \mu)}} \frac{\mathrm{d}\mu}{16\mu^2} = -\frac{f(-\frac{1}{16\rho})}{16\rho^2}.$$

for some $\rho \in \tilde{G}_{2,m}$ hence $\nu = -\frac{1}{16\rho} \in G_{2,m}$ and

$$-\frac{f(-\frac{1}{16\rho})}{16\rho^2} = -16\nu^2 f(\nu).$$

$\square$

Next let us introduce some more notation. Recall that for any $n \geq 0$, $f_n(\lambda) \equiv f_n(s_1, s_2, \lambda)$ is assumed to be of the form

$$f_n(\lambda) = -\frac{1}{\pi_n} \frac{1}{f_{n,2}(s_2, \infty)} f_{n,1}(s_1, \lambda) f_{n,2}(s_2, \lambda).$$

For $j = 1$ and $m \in \mathbb{Z} \setminus \{n\}$ define

$$\frac{f_n(\lambda)}{\sqrt[c]{\chi_p(\lambda)}} = \frac{\sigma_{1,m} - \lambda}{w_{1,m}(\lambda)} \zeta^n_{1,m}(\lambda) \tag{8.34}$$

where

$$\zeta^n_{1,m}(\lambda) = \frac{\mathrm{i}}{w_{1,n}} \Big(\prod_{k \neq n,m} \frac{\sigma_{1,k} - \lambda}{w_{1,k}(\lambda)} \Big) \frac{f_{n,2}(\lambda)/f_{n,2}(\infty)}{\sqrt[c]{\chi_2(\lambda)}/\sqrt[c]{\chi_2(\infty)}}. \tag{8.35}$$

Note that locally uniformly on Ω_n

$$\frac{f_{n,2}(\lambda)}{f_{n,2}(\infty)} = 1 + O(\frac{1}{\lambda}), \qquad \frac{\sqrt[c]{\chi_2(\lambda)}}{\sqrt[c]{\chi_2(\infty)}} = 1 + O(\frac{1}{\lambda}) \quad \text{as } |\lambda| \to \infty.$$

Furthermore, setting $\sigma_{1,n} = \tau_n \in U_{1,n}$ one has $\sigma_{1,n} - \lambda \neq 0 \;\; \forall\, \lambda \in U_{1,m}$, $m \neq n$, and

$$\frac{n-m}{\sigma_{1,n} - \lambda}\Big|_{U_{1,m}} = \frac{1}{\pi} + \ell^2_m \quad \text{as } |m| \to \infty.$$

By Corolary 6.24 one then concludes that for $|m| \to \infty$

$$\mathrm{i}\frac{n-m}{w_{1,n}(\lambda)} \prod_{k \neq n,m} \frac{\sigma_{1,k} - \lambda}{w_{1,k}(\lambda)}\Big|_{U_{1,m}} = \mathrm{i}\frac{n-m}{\sigma_{1,n} - \lambda} \prod_{k \neq m} \frac{\sigma_{1,k} - \lambda}{w_{1,k}(\lambda)}\Big|_{U_{1,m}} = \frac{\mathrm{i}}{\pi} + \ell^2_m$$

locally uniformly on Ω_n. Altogether, we thus have proved that

$$(n-m)\zeta^n_{1,m}(\lambda)\Big|_{U_{1,m}} = \frac{\mathrm{i}}{\pi} + \ell^2_m \quad \text{as } |m| \to \infty \tag{8.36}$$

locally uniformly on Ω_n. In the case $j = 2$, one obtains a similar estimate. For $m \in \mathbb{Z}$ we write

$$\pi_n \frac{f_n(\lambda)}{\sqrt[c]{\chi_p(\lambda)}} = \frac{\sigma_{2,m} + \frac{1}{16\lambda}}{w_{2,m}(\lambda)} \zeta^n_{2,m}(\lambda) \tag{8.37}$$

where

$$\zeta^n_{2,m}(\lambda) = \mathrm{i}\frac{f_{n,1}(\lambda)}{\sqrt[c]{\chi_1(\lambda)}/\sqrt[c]{\chi_1(0)}} \frac{1}{f_{n,2}(\infty)} \prod_{k \neq m} \frac{\sigma_{2,k} + \frac{1}{16\lambda}}{w_{2,k}(\lambda)} \tag{8.38}$$

Here we used that by (6.58), $\sqrt[c]{\chi_1(0)} = \sqrt[c]{\chi_2(\infty)}$. Note that for λ near 0,

$$f_{n,1}(\lambda) = f_{n,1}(0) + O(\lambda), \qquad \sqrt[c]{\chi_1(\lambda)}/\sqrt[c]{\chi_1(0)} = 1 + O(\lambda).$$

One then again concludes from Corollary 6.24 that

$$\zeta^n_{2,m}\Big|_{U_{2,m}} = \mathrm{i}\frac{f_{n,1}(0)}{f_{n,2}(\infty)} + \ell^2_m \quad \text{as } m \to \infty \tag{8.39}$$

locally uniformly on Ω_n.

Proof of Lemma 8.17. Let $n \geq 0$. We first consider the case F^n_1. For $(v, s_1, s_2) \in \Omega_n$ we have by (8.31), (8.34) for $m \neq n$

$$F^n_{1,m} = (n-m) \int_{\Gamma_{1,m}} \frac{\sigma_{1,m} - \lambda}{w_{1,m}(\lambda)} \zeta^n_{1,m}(\lambda)\, \mathrm{d}\lambda. \tag{8.40}$$

By Lemma 6.17, $w_{1,m}(\lambda)$ is analytic on $U'_{1,m} \times V_{v_0}$ with $U'_{1,m}$, V_{v_0} as introduced at the beginning of this section. By Lemma 6.18 and Lemma B.1, $\zeta^n_{1,m}$ is analytic on $U'_{1,m} \times (\Omega_n \cap (V_{v_0} \times \ell^2_{\hat{n}} \times \ell^2))$. Hence the integrand in (8.40) is analytic on $U'_{1,m} \times (\Omega_n \cap (V_{v_0} \times \ell^2_{\hat{n}} \times \ell^2))$ for any $m \neq n$ and $F^n_{1,m}$ is analytic on Ω_n. Moreover, by Lemma 8.19 and (8.36) the right hand side of (8.40) is of the order of $\max_{\lambda \in G_{1,m}} |\sigma_{1,m} - \lambda|$. Since $\sigma_{1,m} = m\pi + \ell^2_m$, $\max_{\lambda \in G_{1,m}} |\tau_{1,m} - \lambda| = |\gamma_{1,m}|/2 = \ell^2_m$ and $\tau_{1,m} = m\pi + \ell^2_m$ (Lemma 3.17) one has locally uniformly $\max_{\lambda\in G_{1,m}}|\sigma_{1,m}-\lambda| = \ell^2_m$. Altogether we thus have proved that $F_1^n : \Omega_n \to \ell^2_{\hat n}$ is locally bounded and, by Theorem A.5, analytic on Ω_n. Now let us turn to F_2^n. On Ω_n we have by (8.32),(8.37) for $m \in \mathbb{Z}$, with the change of variable $\lambda = -\frac{1}{16\mu}$

$$F^n_{2,m} = 16\pi^2_m \int_{\Gamma_{2,m}} \frac{\sigma_{2,m}+\frac{1}{16\lambda}}{w_{2,m}(\lambda)}\zeta^n_{2,m}(\lambda)\,\mathrm{d}\lambda = 16\pi^2_m \int_{\tilde\Gamma_{2,m}} \frac{\sigma^n_{2,m}\mu}{w_{2,m}(-\frac{1}{16\mu})}\zeta^n_{2,m}(-\frac{1}{16\mu})\frac{\mathrm{d}\mu}{16\mu^2}. \tag{8.41}$$

Arguing as in the case of $F^n_{1,m}$ one sees that $F^n_{2,m}$ is analytic on Ω_n. Moreover by Lemma 8.19 and (8.39) one concludes that the right hand side of (8.41) is of the order $\max_{\lambda\in G_{2,m}}\big(\pi^2_m|\lambda|^2|\sigma_{2,m}+\frac{1}{16\lambda}|\big)$. Since $\sigma_{2,m} = m\pi+\ell^2_m$, $\tau_{2,m} = (\lambda^+_{2,m}+\lambda^-_{2,m})/2 = m\pi+\ell^2_m$, $\lambda^+_{2,m}-\lambda^-_{2,m} = \ell^2_m$ one sees that $-\frac{1}{16\lambda} = m\pi+\ell^2_m$ for $\lambda \in G_{2,m}$ and $\max_{\lambda\in G_{2,m}}\pi^2_m|\lambda|^2|\sigma_{2,m}+\frac{1}{16\lambda}| = \ell^2_m$. Altogether we thus have proved that $F_2^n : \Omega_n \to \ell^2$ is locally bounded and, by Theorem A.5, analytic on Ω_n. Next we prove that $F^n_{1,m}$, $F^n_{2,m}$ are real valued on $H^1_r \times \ell^2_{\mathbb{R},\hat n} \times \ell^2_{\mathbb{R}}$ where

$$\ell^2_{\mathbb{R},\hat n} := \ell^2(\mathbb{Z}\setminus\{n\},\ \mathbb{R}).$$

Recall that for $v \in H^1_r$, $\lambda^{\pm}_{j,k} \in \mathbb{R}$ for any $j = 1,2$, $k \in \mathbb{Z}$. Furthermore for any $\lambda \in \mathbb{R}\setminus G_{j,k}$ $(k \in \mathbb{Z})$, $w_{j,k}(\lambda) \in \mathbb{R}$ whereas for $\lambda \in G_{j,k}$, $\mathrm{i}w_{j,k}(\lambda) \in \mathbb{R}$. It then follows that in the case $\lambda^+_{j,m} = \lambda^-_{j,m}$, $F^n_{j,m}$ are real valued on $H^1_r \times \ell^2_{\mathbb{R},\hat n} \times \ell^2_{\mathbb{R}}$ by Cauchy's integral and in the case $\lambda^-_{j,m} < \lambda^+_{j,m}$ by deforming the contour $\Gamma_{j,m}$ to the interval $G_{j,m} \subset \mathbb{R}$. To establish the statement on the uniform bounds for F^n with respect to n, it remains to consider n large. Given $v \in \hat W$ and $(s_1,s_2) \in \ell^2\times\ell^2$, let $k_0 \geq 1$ be such that

$$U_{1,k} = D_k, \quad U_{2,k} = -D_{-k} \quad \forall\ |k| \geq k_0$$

and

$$|s_{j,k}| < \pi/4 \quad \forall\ |k| \geq k_0, \quad j = 1,2.$$

Then for any $|k| \geq k_0$, $\sigma_{j,k} = k\pi + s_{j,k}$, $j = 1,2$ satisfy

$$\sigma_{1,k} \in U_{1,k}, \quad -\frac{1}{16\sigma_{2,k}} \in U_{2,k}.$$

According to (I-2) and (I-3) (cf Section 6.2), for any $|k| \geq k_0$

$$|\sigma_{1,k}-\lambda| \geq |k-m|/c \quad \forall\ \lambda \in U_{1,m},\ \forall\ m \neq k \tag{8.42}$$

$$|\sigma_{2,k}+\frac{1}{16\lambda}| \geq |k-m|/c \quad \forall\ \lambda \in U_{2,m},\ \forall\ m \neq k. \tag{8.43}$$

To estimate $F^n_{1,m}$, write the integrand in (8.40) in the form

$$(n-m)\frac{\sigma_{1,m}-\lambda}{w_{1,m}(\lambda)}\zeta^n_{1,m}(\lambda) = \frac{\sigma_{1,m}-\lambda}{w_{1,m}(\lambda)}\frac{n-m}{\sigma_{1,n}-\lambda}\mathrm{i}\zeta_{1,m}(\lambda) \tag{8.44}$$

where

$$\zeta_{1,m}(\lambda) := \Big(\prod_{k\neq m}\frac{\sigma_{1,k}-\lambda}{w_{1,k}(\lambda)}\Big)\frac{f_{n,2}(\lambda)/f_{n,2}(\infty)}{\sqrt[c]{\chi_2(\lambda)}/\sqrt[c]{\chi_2(\infty)}}. \tag{8.45}$$

Since for $n \geq k_0$, $\big|\frac{n-m}{\sigma_{1,n}-\lambda}\big| \leq C$ for any $\lambda \in U_{1,m}$ with $m \neq n$ and, by Corollary 6.24

$$\sup_{\lambda\in U_{1,m}}|\zeta_{1,m}(\lambda)-1| = \ell^2_m \tag{8.46}$$

locally uniformly on $\Omega \cap (V_v \times \ell^2 \times \ell^2)$, it then follows, arguing as in the first part of the proof, that

$$\sup_{n\in\mathbb{Z}\setminus\{m\}}|F^n_{1,m}| = \ell^2_m$$

locally uniformly on $\Omega \cap (V_v \times \ell^2 \times \ell^2)$. The corresponding estimate for $F^n_{2,m}$ follows from (8.39) and Corollary 6.24. In this way one obtains that $\sup_n|F^n_{2,m}| = \ell^2_m$ locally uniformly on $\Omega\cap(V_v\times\ell^2\times\ell^2)$. $\square$

With the application of the implicit function theorem in mind we now investigate the differential of the maps F^n at a point (v, s_1, s_2). For our purposes it suffices to restrict ourselves to the open subset $\Omega_r := \Omega \cap (H^1_r \times \ell^2_{\mathbb{R}} \times \ell^2_{\mathbb{R}})$. We now compute the differential of F^n with respect to s_1, s_2 at any point of Ω_r. By the analyticity of F^n this is a bounded linear operator

$$Q^n : \ell^2_{\hat{n}} \times \ell^2 \to \ell^2_{\hat{n}} \times \ell^2$$

which is represented by an infinite matrix $\begin{pmatrix} Q^n_{11} & Q^n_{12} \\ Q^n_{21} & Q^n_{22} \end{pmatrix}$ where

$$Q^n_{11,mr} := \frac{\partial F^n_{1,m}}{\partial s_{1,r}} \ (m, r \neq n), \qquad Q^n_{12,mr} := \frac{\partial F^n_{1,m}}{\partial s_{2,r}} \ (m \neq n)$$

$$Q^n_{21,mr} := \frac{\partial F^n_{2,m}}{\partial s_{1,r}} \ (r \neq n), \qquad Q^n_{22,mr} := \frac{\partial F^n_{2,m}}{\partial s_{2,r}} \ (m, r \in \mathbb{Z}).$$

***Lemma* 8.19** *On Ω_r, the matrix elements of Q^n are real and satisfy:*

$$(i)\ 0 \neq Q^n_{11,mm} = 2 + \ell^2_m \ (m \neq n), \quad Q^n_{11,mr} = \frac{\ell^2_m}{|m-r|} \ (r \neq m,\ r, m \neq n), \quad Q^n_{12,mr} = \frac{\ell^2_m}{\langle r \rangle^2} \ (m \neq n)$$

$$(ii)\ 0 \neq Q^n_{22,mm} = 2\pi \frac{f_{n,1}(0)}{f_{n,2}(\infty)} + \ell^2_m, \quad Q^n_{22,mr} = \frac{\ell^2_m}{|m-r|} + \frac{\ell^2_m}{\langle r \rangle} \ (r \neq m), \quad Q^n_{21,mr} = \frac{\ell^2_m}{\langle r \rangle} \ (m, r \in \mathbb{Z}).$$

Proof. By Lemma 8.17, all coefficients of Q^n are real valued on Ω_r. Towards (i) let us first consider $(Q^n_{11})_{mr}$ with $m, r \neq n$. By (8.35) the term $\prod_{k \neq n,m} \frac{\sigma_{1,k} - \lambda}{w_{1,k}(\lambda)}$ of $\zeta^n_{1,m}$ is the only one which depends on s_1. Since for $r \neq n, m$

$$\partial_{s_{1,r}} \Big(\frac{\sigma_{1,r} - \lambda}{w_{1,r}(\lambda)} \Big) = \frac{\sigma_{1,r} - \lambda}{w_{1,r}(\lambda)} \frac{1}{\sigma_{1,r} - \lambda}$$

one has

$$\partial_{s_{1,r}} \zeta^n_{1,m}(\lambda) = \frac{1}{\sigma_{1,r} - \lambda} \zeta^n_{1,m}(\lambda).$$

By (8.40), one then concludes that for $r \neq m, n$

$$Q^n_{11,mr} = (n-m) \int_{\Gamma_{1,m}} \frac{\sigma_{1,m} - \lambda}{w_{1,m}(\lambda)} \frac{1}{\sigma_{1,r} - \lambda} \zeta^n_{1,m}(\lambda) \,\mathrm{d}\lambda. \tag{8.47}$$

By Lemma 8.18 and (8.42), (8.44), (8.46), $\sup_{n \in \mathbb{Z} \setminus \{m,r\}} |Q^n_{11,mr}|$ is of the order

$$\max_{\lambda \in G_{1,m}} |(\sigma_{1,m} - \lambda) \frac{1}{\sigma_{1,r} - \lambda}| = \frac{\ell^2_m}{|m-r|}$$

locally uniformly on Ω_r. For $m = r$, $m \neq n$ the same arguments lead to

$$Q^n_{11,mm} = (n-m) \int_{\Gamma_{1,m}} \frac{\zeta^n_{1,m}(\lambda)}{w_{1,m}(\lambda)} \,\mathrm{d}\lambda.$$

Again using Lemma 8.18 one sees that there exists $\nu \in G_{1,m}$ so that $Q^n_{11,mm} = -2\pi \mathrm{i}(n-m)\zeta^n_{1,m}(\nu)$. By the estimate (8.36) it then follows that $Q^n_{11,mm} = 2 + \ell^2_m$. By inspection of (8.35) one concludes that $Q^n_{11,mm} \neq 0$ for any $m \neq n$. Next let us estimate $Q^n_{12,mr}$. By (8.35), $f_{n,2}(\lambda) = \prod_{k \in \mathbb{Z}} \frac{\sigma_{2,k} + \frac{1}{16\lambda}}{\pi_k}$ and $f_{n,2}(\infty)^{-1}$ are the only terms in $\zeta^n_{1,m}(\lambda)$ which depend on s_2. Since $f_{n,2}(\infty) = \prod_{k \neq n} \frac{\sigma_{2,k}}{w_{2,k}(0)}$, one has for $r \neq n$

$$\partial_{s_{2,r}} (f_{n,2}(\infty))^{-1} = -(f_{n,2}(\infty))^{-1} \frac{1}{\sigma_{2,r}}, \quad \partial_{s_{2,r}} f_{n,2}(\lambda) = f_{n,2}(\lambda) \frac{1}{\sigma_{2,r} + \frac{1}{16\lambda}}$$

implying that

$$Q^n_{12,mr} = (n-m) \int_{\Gamma_{1,m}} \frac{\sigma_{1,m} - \lambda}{w_{1,m}(\lambda)} \Big(\frac{1}{\sigma_{2,r} + \frac{1}{16\lambda}} - \frac{1}{\sigma_{2,r}} \Big) \zeta^n_{1,m}(\lambda) \,\mathrm{d}\lambda. \tag{8.48}$$

For $|m|$ sufficiently large, $U_m = D_m$ implying that $1/16\lambda \in U_{-|m|}$ for any $\lambda \in U_{|m|}$. Hence for such m, $|\sigma_{2,r} + \frac{1}{16\lambda}| \geq \frac{1}{C}\frac{1}{\langle r\rangle}$ for any $\lambda \in U_{1,m}$ and $r \in \mathbb{Z}$. A similar estimate holds for $|r|$ sufficiently large and any $m \in \mathbb{Z}$. Hence for any $\lambda \in G_{1,m}$,

$$|\frac{1}{\sigma_{2,r} + \frac{1}{16\lambda}} - \frac{1}{\sigma_{2,r}}| \leq C\frac{1}{\langle r\rangle^2}.$$

Again using Lemma 8.18 and estimate (8.36) one sees that $\sup_{n\in\mathbb{Z}\setminus\{m\}} |Q^n_{12,mr}| = \frac{1}{\langle r\rangle^2}\ell^2_m$.
(ii) First let us consider $Q^n_{22,mr}$ $(m, r \in \mathbb{Z})$. Arguing as in item (i) one sees that for $m = r$ one has by (8.38) and (8.41)

$$Q^n_{22,mm} = 16\pi^2_m \int_{\Gamma_{2,m}} (1 - \frac{\sigma_{2,m} + \frac{1}{16\lambda}}{\sigma_{2,m}})\frac{\zeta^n_{2,m}(\lambda)}{w_{2,m}(\lambda)}\,\mathrm{d}\lambda = 16\pi^2_m \int_{\tilde{\Gamma}_{2,m}} \frac{\mu}{\sigma_{2,m}}\frac{\zeta^n_{2,m}(-\frac{1}{16\mu})}{w_{2,m}(-\frac{1}{16\mu})}\frac{\mathrm{d}\mu}{16\mu^2}. \tag{8.49}$$

Again using (8.41) and Lemma 8.18 one concludes that there exists $\nu \in G_{2,m}$ so that

$$Q^n_{22,mm} = -2\pi\frac{-\frac{1}{16\nu}}{\sigma_{2,m}}\zeta^n_{2,m}(\nu)16\nu^2 \cdot 16\pi^2_m$$

which by (8.38) does not vanish. Note that since $\nu \in G_{2,m}$

$$\frac{-\frac{1}{16\nu}}{\sigma_{2,m}} = 1 + \ell^2_m, \quad 16\nu^2 \cdot 16\pi^2_m = 1 + \frac{1}{m}\ell^2_m.$$

Hence by (8.39) one has $Q^n_{22,mm} = 2\pi\frac{f_{n,1}(0)}{f_{n,2}(\infty)} + \ell^2_m$ as claimed. Next let us estimate $Q^n_{22,mr}$ for $m \neq r$. Arguing as in the proof of item (i) one sees that

$$Q^n_{22,mr} = 16\pi^2_m \int_{\Gamma_{2,m}} \frac{\sigma_{2,m} + \frac{1}{16\lambda}}{w_{2,m}(\lambda)}(\frac{1}{\sigma_{2,r} + \frac{1}{16\lambda}} - \frac{1}{\sigma_{2,r}})\zeta^n_{2,m}(\lambda)\,\mathrm{d}\lambda.$$

Since for $\lambda \in G_{2,m}$, $-\frac{1}{16\lambda} = m\pi + \ell^2_m$, one has

$$|\frac{1}{\sigma_{2,r} + \frac{1}{16\lambda}}| \leq C\frac{1}{|r-m|}, \quad |\frac{1}{\sigma_{2,r}}| \leq \frac{1}{\langle r\rangle}.$$

By the change of variable $\lambda = -\frac{1}{16\mu}$ one deduces from (8.39) and Lemma 8.18 that

$$\sup_n |Q^n_{22,mr}| = \frac{1}{|m-r|}\ell^2_m + \frac{1}{\langle r\rangle}\ell^2_m$$

as claimed. Finally we estimate $Q^n_{21,mr}$. In this case one has

$$Q^n_{21,mr} = 16\pi^2_m \int_{\Gamma_{2,m}} \frac{\sigma_{2,m} + \frac{1}{16\lambda}}{w_{2,m}(\lambda)}\frac{1}{\sigma_{1,r} + \lambda}\zeta^n_{2,m}(\lambda)\,\mathrm{d}\lambda.$$

Since $|\sigma_{1,r} + \lambda| \geq \frac{1}{C}\langle r\rangle$ for $\lambda \in G_{2,m}$ one gets

$$|\frac{1}{\sigma_{1,r} + \lambda}| \leq C\frac{1}{\langle r\rangle}.$$

Again using Lemma 8.18 and (8.39) one sees that $\sup_n |Q^n_{21,mr}| - \frac{1}{\langle r\rangle}\ell^2_m$ as claimed. □

Lemma 8.20 *For any $(v, s_1, s_2) \in \Omega_r$ and $n \in \mathbb{Z}$, the Jacobian Q^n of F^n with respect to (s_1, s_2) is of the form*

$$Q^n = D^n + K^n : \ell_{\hat{n}} \times \ell^2 \to \ell^2_{\hat{n}} \times \ell^2$$

where D^n is a linear isomorphism in diagonal form and K^n a compact operator.

Proof. Let D^n be the diagonal of Q^n, $D^n = diag((Q^n_{11,mm})_{m\neq n}, (Q^n_{22,mm})_{m\in\mathbb{Z}})$. By the preceding lemma, any diagonal entry is nonzero and $\lim_{|m|\to\infty} Q^n_{11,mm} = 2$, $\lim_{|m|\to\infty} Q^n_{22,mm} = 2\pi\frac{f_{n,1}(0)}{f_{n,2}(\infty)}$. Since $\sigma^n_{j,k} \in \mathbb{C}^*$ and $\sigma^n_{j,k} = \pi k + \ell^2_k$,

$$f_{n,2}(\infty) = \prod_{k\in\mathbb{Z}} \frac{\sigma^n_{2,k}}{\pi_k} \in \mathbb{C}^*, \quad f_{n,1}(0) = \prod_{k\neq n} \frac{\sigma^n_{1,k}}{\pi_k} \in \mathbb{C}^*.$$

Hence $D^n : \ell^2_{\hat{n}} \times \ell^2 \to \ell^2_{\hat{n}} \times \ell^2$ is a linear isomorphism. Moreover, $K^n := Q^n - D^n$ is a bounded linear operator on $\ell^2_{\hat{n}} \times \ell^2$ with vanishing diagonal elements. By the estimates of Lemma 8.19, K^n is Hilbert-Schmidt and hence compact. □

Lemma 8.21 *At any point $(v, s_1, s_2) \in \Omega_r$, the Jacobian Q^n, $n \in \mathbb{Z}$, is one-to-one on $\ell^2_{\hat{n}} \times \ell^2$.*

Proof. Suppose that $Q^n h = 0$ for some $h = (h_1, h_2) \in \ell^2_{\hat{n}} \times \ell^2$. Recall that $f_n(s_1, s_2, \lambda)$ is analytic on $\ell^2_{\hat{n}} \times \ell^2 \times \mathbb{C}^*$. In particular, $\phi_n(\lambda) := \partial_\epsilon|_{\epsilon=0} f_n((s_1, s_2) + \epsilon(h_1, h_2), \lambda)$ is analytic on $\mathbb{C}^*$. By assumption we have for any $m \neq n$

$$0 = \sum_{r\neq n} Q^n_{11,mr} h_{1,r} + \sum_{r\in\mathbb{Z}} Q^n_{12,mr} h_{2,r} = (n-m)\int_{\Gamma_{1,m}} \frac{\phi_n(\lambda)}{\sqrt[c]{\chi_p(\lambda)}}\, \mathrm{d}\lambda$$

and for any $m \in \mathbb{Z}$

$$0 = \sum_{r\in\mathbb{Z}} Q^n_{21,mr} h_{1,r} + \sum_{r\in\mathbb{Z}} Q^n_{22,mr} h_{2,r} = 16\pi^2_m \pi_n \int_{\Gamma_{2,m}} \frac{\phi_n(\lambda)}{\sqrt[c]{\chi_p(\lambda)}}\, \mathrm{d}\lambda.$$

We will use Lemma D.1 (interpolation lemma) to show that $\phi_n \equiv 0$ which will imply that $h = 0$. Since $\phi_n(\lambda)$ is real for $\lambda \in \mathbb{R}^*$, it follows from Lemma 8.16 that $\phi_n(\lambda)$ has roots $\rho_{1,m} \in G_{1,m}$ ($m \neq n$), and $-\frac{1}{16\rho_{2,m}} \in G_{2,m}$ ($m \in \mathbb{Z}$). On the other hand, for $\lambda \in \partial B_N$ or $\lambda \in \partial B_{-N}$ with N sufficiently large, $\sigma_{1,r} - \lambda \neq 0$, $\sigma_{2,r} + \frac{1}{16\lambda} \neq 0$ for any $r \neq 0$. Therefore $\phi_n(\lambda)$ for $\lambda \in \partial B_N$ or $\lambda \in \partial B_{-N}$ can be written as

$$\begin{aligned}\phi_n(\lambda) =& \sum_{r\neq n} \partial_{s_{1,r}}(f_n(\lambda)) h_{1,r} + \sum_{r\in\mathbb{Z}} \partial_{s_{2,r}}(f_n(\lambda)) h_{2,r} \\ =& f_n(\lambda) \sum_{r\neq n} \frac{1}{\sigma_{1,r} - \lambda} h_{1,r} + f_n(\lambda) \sum_{r\in\mathbb{Z}} \Big(\frac{1}{\sigma_{2,r} + \frac{1}{16\lambda}} - \frac{1}{\sigma_{2,r}}\Big) h_{2,r}.\end{aligned}$$

We want to apply Lemma D.1 to

$$\phi(\lambda) := \phi_n(\lambda)(\sigma_{1,n} - \lambda).$$

We now verify that its assumptions are satisfied. First we show that $\sup_{\lambda\in\partial B_N} |\frac{\phi(\lambda)}{\sin(\lambda)}| \to 0$ as $N \to \infty$. By the product representation of $\sin(\lambda)$, $\sin(\lambda) = -\prod_{m\in\mathbb{Z}} \frac{m\pi - \lambda}{\pi_m}$, one gets for $\lambda \in \partial B_N$

$$f_n(\lambda) \cdot \frac{\sigma_{1,n} - \lambda}{\sin(\lambda)} = -\frac{f_{n,2}(\lambda)}{f_{n,2}(\infty)} \frac{f_{n,1}(\lambda)}{\sin(\lambda)} \frac{\sigma_{1,n} - \lambda}{\pi_n} = \frac{f_{n,2}(\lambda)}{f_{n,2}(\infty)} \prod_{k\in\mathbb{Z}} \frac{\sigma_{1,k} - \lambda}{k\pi - \lambda}.$$

It implies that the quotient of $\phi(\lambda) = \phi_n(\lambda)(\sigma_{1,n} - \lambda)$ with $\sin(\lambda)$ can be written as

$$\frac{\phi(\lambda)}{\sin(\lambda)} = \phi_n(\lambda)\frac{\sigma_{1,n} - \lambda}{\sin(\lambda)} = \frac{f_{n,2}(\lambda)}{f_{n,2}(\infty)} \Big(\prod_{k\in\mathbb{Z}} \frac{\sigma_{1,k} - \lambda}{k\pi - \lambda}\Big) \cdot \sum_{r\neq n} \Big(\frac{1}{\sigma_{1,r} - \lambda} h_{1,r} + \sum_{r\in\mathbb{Z}} \Big(\frac{1}{\sigma_{2,r} + \frac{1}{16\lambda}} - \frac{1}{\sigma_{2,r}}\Big) h_{2,r}.$$

By Lemma B.5

$$\sup_{\lambda\in\partial B_N} \Big|\prod_{k\in\mathbb{Z}} \frac{\sigma_{1,k} - \lambda}{k\pi - \lambda}\Big| = O(1) \quad \text{as } N \to \infty.$$

Since $f_{n,2}$ is analytic near $\lambda = \infty$ and $f_{n,2}(\infty) \neq 0$ one has

$$\sup_{\lambda\in\partial B_N} \Big|\sum_{r\neq n} \frac{1}{\sigma_{1,r} - \lambda} h_{1,r}\Big| \leq C\Big(\sum_{|r|\leq N/2} \frac{1}{(r-N)^2}\Big)^{1/2} + C\Big(\sum_{|r|>N/2} |h_{1,r}|^2\Big)^{1/2} = o(1) \quad \text{as } N \to \infty$$

and, since $\frac{1}{\sigma_{2,r}+\frac{1}{16\lambda}} - \frac{1}{\sigma_{2,r}} = \frac{\frac{1}{16\lambda}}{(\sigma_{2,r}+\frac{1}{16\lambda})\sigma_{2,r}}$ one has

$$\sup_{\lambda\in\partial B_N} \Big|\sum_{r\in\mathbb{Z}} \Big(\frac{1}{\sigma_{2,r}+\frac{1}{16\lambda}} - \frac{1}{\sigma_{2,r}}\Big)h_{2,r}\Big| \le C \sup_{\lambda\in\partial B_N} \frac{1}{16|\lambda|}\Big(\sum_{r\in\mathbb{Z}} \frac{1}{\langle r\rangle^2} h_{2,r}\Big) \to 0 \quad \text{as } N\to\infty.$$

Altogether it thus follows that

$$\sup_{\lambda\in\partial B_N} \Big|\frac{\phi(\lambda)}{\sin(\lambda)}\Big| = o(1) \quad \text{as } N\to\infty.$$

Similarly we estimate

$$\begin{aligned}
\sup_{\lambda\in\partial B_{-N}} \Big|\frac{\phi(\lambda)}{\sin(-\frac{1}{16\lambda})}\Big| &= \sup_{\mu\in\partial B_N} \Big|\frac{\phi(-\frac{1}{16\mu})}{\sin(\mu)}\Big| \\
&= \sup_{\mu\in\partial B_N} \Big(\frac{|\sigma_{1,n}+\frac{1}{16\mu}|}{f_{n,2}(\infty)}\Big|\frac{f_{n,2}(-\frac{1}{16\mu})}{\sin(\mu)}\Big|\Big|f_{n,1}(-\frac{1}{16\mu})\Big|\Big) \\
&\quad \cdot \Big(\Big|\sum_{r\ne n} \frac{1}{\sigma_{1,r}+\frac{1}{16\mu}}h_{1,r}\Big| + \Big|\sum_{r\in\mathbb{Z}} \Big(\frac{1}{\sigma_{2,r}-\mu} - \frac{1}{\sigma_{2,r}}\Big)h_{2,r}\Big|\Big).
\end{aligned}$$

Since $|\frac{1}{\sigma_{1,r}+\frac{1}{16\mu}}| \le C\frac{1}{\langle r\rangle}$ we get

$$\sum_{r\ne n}\Big|\frac{1}{\sigma_{1,r}+\frac{1}{16\mu}}h_{1,r}\Big| \le \Big(\sum_{r\ne n}\frac{1}{|\sigma_{1,r}+\frac{1}{16\mu}|^2}\Big)^{1/2}\Big(\sum_{r\in\mathbb{Z}}|h_{1,r}|^2\Big)^{1/2} = O(1)$$

and since

$$\Big|\frac{1}{\sigma_{2,r}-\mu} - \frac{1}{\sigma_{2,r}}\Big| \le \frac{1}{|\sigma_{2,r}-\mu|} + \frac{1}{|\sigma_{2,r}|} \le C\Big(\frac{1}{|r-(N+1/2)|} + \frac{1}{|r+(N+1/2)|} + \frac{1}{\langle r\rangle}\Big)$$

it follows that

$$\sum_{r\in\mathbb{Z}}\Big|\frac{1}{\sigma_{2,r}-\mu} - \frac{1}{\sigma_{2,r}}\Big||h_{2,r}| \le C\Big(\sum_r |h_{2,r}|^2\Big)^{1/2}.$$

Altogether this shows

$$\sup_{\lambda\in\partial B_{-N}} \Big|\frac{\phi(\lambda)}{\sin(\lambda)}\Big| = O(1) \text{ and hence } o(N) \quad \text{as } N\to\infty.$$

Hence we can apply Lemma D.1 to $\phi(\lambda)$. Since $\phi(\rho_{1,m}) = 0$ $(m\ne n)$, $\phi(-(16\rho_{2,m})^{-1}) = 0$ $(m\in\mathbb{Z})$, and $\rho_{j,m} = m\pi + \ell^2_r$ it then follows that $\phi\equiv 0$ and hence $\phi_n\equiv 0$, or for any $\lambda\in\mathbb{C}^*$

$$0 = \sum_{r\ne n}\frac{f_n(\lambda)}{\sigma_{1,r}-\lambda}h_{1,r} + \sum_{r\in\mathbb{Z}}\frac{-\frac{1}{16\lambda}}{(\sigma_{2,r}+\frac{1}{16\lambda})\sigma_{2,r}}f_n(\lambda)h_{2,r}. \tag{8.50}$$

Evaluating the right hand side of (8.50) at $\lambda = \sigma_{1,m}$ for $m\ne n$ one obtains $0 = \dot f_n(\sigma_{1,m})h_{1,m}$. Since $\sigma_{1,m}$ is a simple root of $f_n(\lambda)$, $\dot f_n(\sigma_{1,m}) \ne 0$ and hence $h_{1,m} = 0$. Similarly, evaluating the right hand side of (8.50) at $\kappa_{2,m} = -(16\sigma_{2,m})^{-1}$ for any given $m\in\mathbb{Z}$ one obtains $0 = \dot f_n(\kappa_{2,m})h_{2,m}$, yielding $h_{2,m} = 0$. Altogether we have shown that $h = 0$ and hence Q^n is one-to-one. □

The latter two lemmas together with the Fredholm alternative yield

***Corollary* 8.22** *At any point in Ω_r, the Jacobian Q^n is a linear isomorphism of $\ell^2_{\hat n}\times\ell^2$.*

Lemma 8.17 and Corollary 8.22 allow to apply the implicit function theorem to any particular solution $s^n = (s^n_1, s^n_2)$ of $F^n(v, s_1, s_2) = 0$ in $\Omega_{r,n} = \Omega_r\cap\Omega_n$.

***Proposition* 8.23** *For any $n\ge 0$, there exists a real analytic map*

$$s^n = (s^n_1, s^n_2) : H^1_r \to \ell^2_{\hat n}\times\ell^2$$

with graph in $\Omega_{r,n}$ such that $F^n(v, s^n(v)) = 0$ for any $v\in H^1_r$. Actually, for any $v\in H^1_r$, $\sigma^n_{1,m}\in G_{1,m}(v)$, for any $m\ne n$ and $\sigma^n_{2,m}\in G_{2,m}(v)$ for any $m\in\mathbb{Z}$, The map s^n is unique within the class of all such real analytic maps with graph in $\Omega_{r,n}$. For $v = 0$,

$$\sigma^n_{1,k} = k\pi + s^n_{1,k} = \tau_{1,k}\ (k\ne n), \quad \sigma^n_{2,k} = k\pi + s^n_{2,k} = \tau_{2,k}\ (k\in\mathbb{Z}). \tag{8.51}$$

Proof. First we note that any solution of $F^n(v,s)=0$ in $\Omega_{r,n}$ satisfies

$$\sigma_{1,m}=m\pi+s_{1,m}\in G_{1,m}(v)\ \ \forall\, m\neq n,\quad \sigma_{2,m}=m\pi+s_{2,m}\in G_{2,m}(v)\ \ \forall\, m\in\mathbb{Z} \tag{8.52}$$

where $s=(s_1,s_2)\in\ell^2_{\hat{n}}\times\ell^2$. Indeed, by Lemma 8.16, for any $m\neq n$, $f_{1,n}(s_1,\lambda)$ has a root $\rho_{1,m}\in G_{1,m}$ and for any $m\in\mathbb{Z}$, $f_{2,n}(s_2,\lambda)$ has a root $-(16\rho_{2,m})^{-1}\in G_{2,m}$. By assumption, $(v,s)\in\Omega_{r,n}$ and hence $\sigma_{1,m}\in U_{1,m}$ $(m\neq n)$, $-(16\sigma_{2,m})^{-1}\in U_{2,m}$ $(m\in\mathbb{Z})$. Since $\sigma_{1,m}$ $(m\neq n)$, $-(16\sigma_{2,m})^{-1}$ $(m\in\mathbb{Z})$ are the only roots of $f_n(v,s)$ and $U_{j,m}$ are pairwise disjoint it follows that $\rho_{1,m}=\sigma_{1,m}$ $(m\neq n)$ and $\rho_{2,m}=\sigma_{2,m}$ $(m\in\mathbb{Z})$. By Lemma 8.17, Corollary 8.22, and the implicit function theorem, any given solution $(v^0,s^0)\in\Omega_{r,n}$ of $F^n(v,s)=0$ can be uniquely extended locally so that s is given as a real analytic function of v. We claim that by the continuation method, this local solution can be extended along any path from v^0 to any given point in H^1_r since by Corollary 8.22

$$\partial_s F:\ell^2_{\hat{n}}\times\ell^2\to\ell^2_{\hat{n}}\times\ell^2$$

is a linear isomorphism at each point in $\Omega_{r,n}$. Indeed, let $(v^k,s(v^k))_{k\geq1}$ be any sequence in $\Omega_{r,n}$ with $F^n(v^k,s(v^k))=0$ for any $k\geq1$ and $v=\lim_{k\to\infty}v^k$ in H^1_r. By Proposition 6.3, the endpoints of $G_{j,m}(v^k)$ converge to the endpoints of $G_{j,m}(v)$. As for any $j=1,2$, $m\in\mathbb{Z}$, and $k\geq1$, $G_{j,m}(v^k)$ and $G_{j,m}(v)$ are compact intervals there exists a subsequence of $(v^k)_{k\geq1}$, which we again denote by $(v^k)_{k\geq1}$ such that $(\sigma_{j,m}(v^k))_{k\geq1}$ converges. Its limit, denoted by $\sigma_{j,m}(v)$, then satisfies $\sigma_{j,m}(v)\in G_{j,m}(v)$ and $F^n_{j,m}(v,s(v))=0$ where $s(v)=(s_1(v),s_2(v))$ and $s_{1,k}(v)=\sigma_{1,k}(v)-k\pi$ $(k\neq n)$, $s_{2,k}(v)=\sigma_{2,k}(v)-k\pi$ $(k\in\mathbb{Z})$. Hence $(v,s(v))\in\Omega_{r,n}$ and we can apply the implicit function theorem to F^n at $(v,s(v))$. This shows that the continuity method applies. Since H^1_r is simply connected, any particular solution $(v^0,s^0)\in\Omega_{r,n}$ of $F^n(v,s)=0$ thus extends uniquely and globally to a real analytic map $s^n:H^1_r\to\ell^2_{\hat{n}}\times\ell^2$ with graph in $\Omega_{r,n}$, satisfying $F^n(v,s^n(v))=0$ everywhere. At $v=0$, one verifies in a straightforward way with Cauchy's formula that a solution of $F^n(0,s^n)=0$ is given by

$$s^n_{1,k}:=\tau_{1,k}-k\pi\ (k\neq n),\quad s^n_{2,k}:=\tau_{2,k}-k\pi\ (k\in\mathbb{Z}).$$

Clearly $(0,s^n)\in\Omega_{r,n}$. Note that this solution is also unique since $G_{j,m}=\{\tau_{j,m}(0)\}$ for any $j=1,2$ and $m\in\mathbb{Z}$. We thus have established (8.51) and shown that there is exactly one such real analytic map. □

We now turn to the question of analytically extending the maps s^n to a common complex neighborhood of H^1_r.

Lemma 8.24 *All the real analytic maps $s^n:H^1_r\to\ell^2_{\hat{n}}\times\ell^2$ of Proposition (8.31) extend to a complex neighborhood $W\subset\hat{W}$ of H^1_r which is independent of n so that for any potential $v\in H^1_r$ and any $n\in\mathbb{Z}$, the restriction of the solution s^n to $W\cap V_v$ satisfies $\sigma^n_{j,m}\in U_{j,m}$ (with $\sigma^n_{1,n}=\tau_{1,n}$) where $U_{j,m}$, $j=1,2$, $m\in\mathbb{Z}$, are neighborhoods in $\mathbb{C}^*$, defined in terms of isolating neighborhoods U_m, $m\in\mathbb{Z}$, for V_v.*

Proof. Recall that by Proposition 8.23, there exists for each $n\geq0$ and $v\in H^1_r$ a solution $s^n(v)$ so that $F^n(v,s^n(v))=0$. Furthermore, by Corollary 8.22, $Q^n=\partial_sF^n:\ell^2_{\hat{n}}\times\ell^2\to\ell^2_{\hat{n}}\times\ell^2$ at $(v,s^n(v))$ is invertible and hence by the implicit function theorem there exists a (simply connected) complex neighborhood $V_{v,n}\subset\Omega$ of v so that the solution s^n extends as a real analytic map to $V_{v,n}$. We now prove estimates of the operator norm $\|Q^n(v,s)^{-1}\|$ of the inverse $Q^n(v,s)^{-1}$ which will allow for any $v\in H^1_r$ to chose $V_{v,n}$ independently of n. As a first step we prove in Lemma 8.25 below that on

$$\Omega_{r,*}:=\{\,(v,s)\in\Omega_r:\ \sigma_{1,k}\in G_{1,k}(v),\ (-16\sigma_{2,k})^{-1}\in G_{2,k}\ \ \forall\, k\in\mathbb{Z}\,\}$$

the operator norm $\|Q^n(v,s)^{-1}\|$ of $Q^n(v,s)^{-1}$ is bounded uniformly in n, locally uniformly in $v\in H^1_r$ and for any given $v\in H^1_r$ uniformly in s. As before, $\sigma_{j,k}=k\pi+s_{j,k}$ for any $j=1,2$ and $k\in\mathbb{Z}$. We now extend Lemma 8.25 to a complex neighborhood. By Lemma 8.17, the maps

$$F^n=(F^n_1,F^n_2):\Omega\to\ell^2_{\hat{n}}\times\ell^2,(v,s)\mapsto F^n(v,s),\ s=(s_1,s_2)$$

are real analytic, uniformly bounded with respect to n, and locally uniformly bounded on the (simply connected) neighborhood $\Omega\subset\hat{W}^2\times\ell^2$. Then the maps

$$Q^n=\partial_sF^n:\Omega\to\mathcal{L}(\ell^2_{\hat{n}}\times\ell^2)$$

are real analytic as well. By Cauchy's estimate, Q^n is uniformly bounded with respect to n and locally uniformly bounded on Ω. Hence $\mathcal{L}(\ell^2_{\hat{n}}\times\ell^2)$ denotes the space of bounded linear operators on $\ell^2_{\hat{n}}\times\ell^2$.

It then follows again by Cauchy's estimate that the variation δQ^n of Q^n with respect to v and s can be kept as small as needed by restricting oneself to a sufficiently small neighborhood of any given point $(v,s) \in \Omega$. Representing $(Q+\delta Q)^{-1}$ by its Neumann series one obtains the standard estimate $\|(Q+\delta Q)^{-1}\| \le 2\|Q^{-1}\|$ for any δQ with $\|\delta Q\| \le \frac{1}{2\|Q^{-1}\|}$ for $Q = Q^n$ with $n \ge 0$. Hence if for a given $(v,s) \in \Omega$, $\|(Q^n)^{-1}\|$ can be bounded uniformly in n, the same holds for elements in a sufficiently small neighborhood of (v,s) in Ω. By Lemma 8.25 it then follows that $\sup_{n\ge 0}\|(Q^n)^{-1}\|$ can be bounded locally uniformly on a simply connected complex neighborhood of $\Omega_{r,*}$, contained in

$$\bigcup_{v\in H^1_r} V_v \times \{\, (s_1,s_2) \in \ell^2 \times \ell^2 \, : \, \sigma_{1,k} \in U_{1,k}, \; (-16\sigma_{2,k})^{-1} \in U_{2,k} \;\; \forall\, k \in \mathbb{Z} \,\}.$$

Using again the continuation method, the solution s^n can then be extended by the implicit function theorem to a complex neighborhood $W \subset \hat{W}$ of H^1_r which is independent of $n \ge 0$ so that for any $v \in V_{v_0} \cap W$ with $v_0 \in H^1_r$, the sequences $(\sigma^n_{j,m}(v))_{m\in\mathbb{Z}}$ given by

$$\sigma^n_{j,m}(v) = m\pi + s^n_{j,m}(v) \quad \forall\, j = 1,2, \;\; \forall\, m \in \mathbb{Z}$$

satisfy

$$\sigma^n_{1,j}(v) \in U_{1,m}(v_0), \quad (-16\sigma^n_{2,m}(v))^{-1} \in U_{2,m}(v_0) \quad \forall\, m \in \mathbb{Z}.$$

□

The following lemma was used in the proof of Lemma 8.24. Recall that

$$\Omega_{r,*} = \{\, (v,s) \in \Omega \, : \, \sigma_{1,k} \in G_{1,k}(v), \; (-16\sigma_{2,k})^{-1} \in G_{2,k}(v) \;\; \forall\, k \in \mathbb{Z} \,\}$$

***Lemma* 8.25** *For any $(v,s) \in \Omega_{r,*}$ $\|(Q^n(v,s))^{-1}\|$ is uniformly bounded with respect to n, locally uniformly with respect to (v,s) and for any given $v \in H^1_r$, uniformly with respect to s with $(v,s) \in \Omega_{r,*}$.*

Proof. We begin by investigating the asymptotics of $Q^n(v,s)$ as $|n| \to \infty$ for $(v,s) \in \Omega_{r,*}$. To this end we consider the infinite matrices Q^n_{11}, Q^n_{12}, Q^n_{21}, and Q^n_{22} in the matrix representation of Q^n

$$Q^n = \begin{pmatrix} Q^n_{11} & Q^n_{12} \\ Q^n_{21} & Q^n_{22} \end{pmatrix}$$

individually. First let us consider $Q^n_{11,mr} = \frac{\partial F^n_{1,m}}{\partial s_{1,r}}$ $(m, r \neq n)$. By (8.47) and (8.44) - (8.46), for $r \neq m$

$$Q^n_{11,mr} = (n-m)\int_{\Gamma_{1,m}} \frac{\sigma_{1,m}-\lambda}{w_{1,m}(\lambda)} \frac{1}{\sigma_{1.r}-\lambda} \zeta^n_{1,m}(\lambda)\,\mathrm{d}\lambda$$

which by (8.44) - (8.46) can be written as

$$Q^n_{11,mr} = \frac{\mathrm{i}}{\pi}\int_{\Gamma_{1,m}} \frac{\sigma_{1,m}-\lambda}{\sigma_{1,r}-\lambda} \frac{n\pi - m\pi}{\sigma_{1,n}-\lambda} \frac{\zeta_{1,m}(\lambda)}{w_{1,m}(\lambda)}\,\mathrm{d}\lambda$$

where

$$\zeta_{1,m}(\lambda) = \prod_{k\neq m} \frac{\sigma_{1,k}-\lambda}{w_{1,k}(\lambda)} \frac{f_2(\lambda)/f_2(\infty)}{\sqrt[c]{\chi_2(\lambda)}/\sqrt[c]{\chi_2(\infty)}}, \quad \sigma_{1,n} \in U_{1,n}.$$

Here we have dropped the subindex n in $f_{n,2}(\lambda)$ and simply write $f_2(\lambda) = \prod_{k\in\mathbb{Z}} \frac{\sigma_{2,k}+\frac{1}{16\lambda}}{\pi_k}$ and $f_2(\infty) = \prod_{k\in\mathbb{Z}} \frac{\sigma_{2,k}}{\pi_k}$. Since $\sigma_{1,n} \in U_{1,n}$ we have

$$\frac{n\pi - m\pi}{\sigma_{1,n}-\lambda} = 1 + \frac{(n\pi - \sigma_{1,n}) + (\lambda - m\pi)}{\sigma_{1,n}-\lambda} = 1 + O(\frac{1}{n-m}).$$

It implies that $Q^*_{11,mr} := \lim_{n\to\infty} Q^n_{11,mr}$ exists and

$$Q_{11,mr}(v,s) = \frac{\mathrm{i}}{\pi}\int_{\Gamma_{1,m}} \frac{\sigma_{1,m}-\lambda}{\sigma_{1,r}-\lambda} \frac{\zeta_{1,m}(\lambda)}{w_{1,m}(\lambda)}\,\mathrm{d}\lambda.$$

Similarly, for $m = r$ one has

$$Q^n_{11,mm} = \frac{\mathrm{i}}{\pi}\int_{\Gamma_{1,m}} \frac{n\pi - m\pi}{\sigma_{1,n}-\lambda} \frac{\zeta_{1,m}(\lambda)}{w_{1,m}(\lambda)}\,\mathrm{d}\lambda.$$

Hence $Q^*_{11,mm} = \lim_{n\to\infty} Q^n_{11,mm}$ exists and

$$Q^*_{11,mm} = \frac{\mathrm{i}}{\pi}\int_{\Gamma_{1,m}} \frac{\zeta_{1,m}(\lambda)}{w_{1,m}(\lambda)}\,\mathrm{d}\lambda.$$

By inspection of $\zeta_{1,m}(\lambda)$ one concludes that $Q^*_{11,mm} \neq 0$ for any $m \in \mathbb{Z}$. Next let us consider $Q^n_{12,mr}$. By (8.48) for any $m, r \neq n$,

$$Q^n_{12,mr} = (n-m)\int_{\Gamma_{1,m}} \frac{\sigma_{1,m}-\lambda}{w_{1,m}(\lambda)}\Big(\frac{1}{\sigma_{2,r}+\frac{1}{16\lambda}} - \frac{1}{\sigma_{2,r}}\Big)\zeta^n_{1,m}(\lambda)\,\mathrm{d}\lambda$$

which by (8.44) - (8.46) can be written as

$$Q^n_{12,mr} = \frac{\mathrm{i}}{\pi}\int_{\Gamma_{1,m}} \frac{\sigma_{1,m}-\lambda}{w_{1,m}(\lambda)}\frac{n\pi - m\pi}{\sigma_{1,n}-\lambda}\Big(\frac{1}{\sigma_{2,r}+\frac{1}{16\lambda}} - \frac{1}{\sigma_{2,r}}\Big)\zeta_{1,m}(\lambda)\,\mathrm{d}\lambda.$$

As above we conclude that for any $m, r \in \mathbb{Z}$, $Q^*_{12,mr} := \lim_{n\to\infty} Q^n_{12,mr}$ exists and

$$Q^*_{12,mr} = \frac{\mathrm{i}}{\pi}\int_{\Gamma_{1,m}} \frac{\sigma_{1,m}-\lambda}{w_{1,m}(\lambda)}\Big(\frac{1}{\sigma_{2,r}+\frac{1}{16\lambda}} - \frac{1}{\sigma_{2,r}}\Big)\zeta_{1,m}(\lambda)\,\mathrm{d}\lambda.$$

Next let us consider $Q_{22,mr}$. For $m = r$, one has by (8.49)

$$Q^n_{22,mm} = 16\pi_m^2\int_{\Gamma_{2,m}} \Big(1 - \frac{\sigma_{2,m}+\frac{1}{16\lambda}}{\sigma_{2,m}}\Big)\frac{\zeta^n_{2,m}(\lambda)}{w_{2,m}(\lambda}\,\mathrm{d}\lambda = 16\pi_m^2\int_{\tilde\Gamma_{2,m}} \frac{\mu}{\sigma_{2,m}}\frac{\zeta^n_{2,m}(-\frac{1}{16\mu})}{w_{2,m}(-\frac{1}{16\mu})}\frac{\mathrm{d}\mu}{16\mu^2}$$

which by (8.38) can be written as

$$Q^n_{22,mm} = \mathrm{i}\int_{\tilde\Gamma_{2,m}} \frac{\mu}{\sigma_{2,m}}\frac{\pi_m^2}{\mu^2}\frac{\pi_n}{\sigma_{1,n}-\frac{1}{16\mu}}\frac{\zeta_{2,m}(-\frac{1}{16\mu})}{w_{2,m}(-\frac{1}{16\mu})}\,\mathrm{d}\mu$$

where

$$\zeta_{2,m}(\lambda) = \Big(\prod_{k\in\mathbb{Z}}\frac{\sigma_{1,k}-\lambda}{\pi_k}\Big)\frac{1}{\sqrt[c]{\chi_1(\lambda)}/\sqrt[c]{\chi_1(0)}}\frac{1}{f_2(\infty)}\prod_{k\neq m}\frac{\sigma_{2,k}+\frac{1}{16\lambda}}{w_{2,k}(\lambda)}.$$

Since $\frac{\pi_n}{\sigma_{1,n}-\frac{1}{16\mu}} = 1 + O(\frac{1}{n})$ it follows that $Q^*_{22,mm} := \lim_{n\to\infty} Q^n_{22,mm}$ exists and

$$Q^*_{22,mm} = \mathrm{i}\int_{\tilde\Gamma_{2,m}} \frac{\mu}{\sigma_{2,m}}\frac{\pi_m^2}{\mu^2}\frac{\zeta_{2,m}(-\frac{1}{16\mu})}{w_{2,m}(-\frac{1}{16\mu})}\,\mathrm{d}\mu \neq 0.$$

Arguing as in the proof of Lemma 8.19 one sees that

$$Q^*_{22,mm} = 2\pi\prod_{k\in\mathbb{Z}}\frac{\sigma_{1,k}}{\sigma_{2,k}} + \ell^2_m. \tag{8.53}$$

Similarly, for $m \neq r$ one has

$$\begin{aligned} Q^n_{22,mr} =& 16\pi_m^2\int_{\Gamma_{2,m}} \frac{\sigma_{2,m}+\frac{1}{16\lambda}}{w_{2,m}(\lambda)}\Big(\frac{1}{\sigma_{2,r}+\frac{1}{16\lambda}} - \frac{1}{\sigma_{2,r}}\Big)\zeta^n_{2,m}(\lambda)\,\mathrm{d}\lambda \\ =& \mathrm{i}\int_{\tilde\Gamma_{2,m}} \frac{\sigma_{2,m}-\mu}{w_{2,m}(-\frac{1}{16\mu})}\Big(\frac{1}{\sigma_{2,r}-\mu} - \frac{1}{\sigma_{2,r}}\Big)\frac{\pi_n}{\sigma_{1,n}-\frac{1}{16\mu}}\zeta_{2,m}\Big(-\frac{1}{16\mu}\Big)\frac{\pi_m^2}{\mu^2}\,\mathrm{d}\mu. \end{aligned}$$

It implies that $Q^*_{22,mr} = \lim_{n\to\infty} Q^n_{22,mr}$ exists and

$$Q^*_{22,mr} = \mathrm{i}\int_{\tilde\Gamma_{2,m}} \frac{\sigma_{2,m}-\mu}{w_{2,m}(-\frac{1}{16\mu})}\Big(\frac{1}{\sigma_{2,r}-\mu} - \frac{1}{\sigma_{2,r}}\Big)\zeta_{2,m}\Big(-\frac{1}{16\mu}\Big)\frac{\pi_m^2}{\mu^2}\,\mathrm{d}\mu.$$

Finally, for $m, r \in \mathbb{Z}$ one has

$$\begin{aligned} Q^n_{21,mr} =& 16\pi_m^2\int_{\Gamma_{2,m}} \frac{\sigma_{2,m}+\frac{1}{16\lambda}}{w_{2,m}(\lambda)}\frac{1}{\sigma_{1,r}+\lambda}\zeta^n_{2,m}(\lambda)\,\mathrm{d}\lambda \\ =& \mathrm{i}\int_{\tilde\Gamma_{2,m}} \frac{\sigma^n_{2,m}\mu}{w_{2,m}(-\frac{1}{16\mu})}\frac{1}{\sigma_{1,r}-\frac{1}{16\mu}}\frac{\pi_n}{\sigma_{1,n}-\frac{1}{16\mu}}\zeta_{2,m}\Big(-\frac{1}{16\mu}\Big)\frac{\pi_m^2}{\mu^2}\,\mathrm{d}\mu. \end{aligned}$$

Again, the limit $Q^*_{21,mr} := \lim_{n\to\infty} Q^n_{21,mr}$ exists and

$$Q^*_{21,mr} = \mathrm{i}\int_{\tilde{\Gamma}_{2,m}} \frac{\sigma^n_{2,m}\mu}{w_{2,m}(-\frac{1}{16\mu})} \frac{1}{\sigma_{1,r}-\frac{1}{16\mu}} \zeta_{2,m}(-\frac{1}{16\mu})\frac{\pi_m^2}{\mu^2}\,\mathrm{d}\mu.$$

By Lemma 8.19 and its proof one sees that the coefficients $Q^*_{jj',mr}$ $(1\le j,j'\le 2,\ m,r\in\mathbb{Z})$ satisfy the same asymptotic estimates as the ones for $Q^n_{jj',mr}$ of Lemma 8.19 except the one for $Q^*_{22,mm}$ which is given by (8.53). Hence

$$Q^* = \begin{pmatrix} Q^*_{11} & Q^*_{12} \\ Q^*_{2} & Q^*_{22}\end{pmatrix} : \ell^2\times\ell^2 \to \ell^2\times\ell^2$$

defines a bounded linear operator on $\ell^2\times\ell^2$. By a slight abuse of notation we now view Q^n as an operator on $\ell^2\times\ell^2$ by setting $Q^n_{11,nn}=2$ and $Q^n_{11,nm}=0$, $Q^n_{11,mn}=0$ for any $m\in\mathbb{Z}\setminus\{n\}$ as well as $Q^n_{12,nm}=0$ for any $m\in\mathbb{Z}$. We claim that $Q^n\to Q^*$ in operator norm, locally uniformly on $\Omega_{r,*}$. To see this, split Q^* into its diagonal part D^* and its off-diagonal part K^*, $Q^*=D^*+K^*$. Arguing as in the proof of Lemma 8.19, one sees that the following holds

$$\begin{aligned}
0\ne D^*_{11,m} :=& Q^*_{11,mm} = 2+\ell^2_m, \quad \forall\, m\in\mathbb{Z}\\
0\ne D^*_{22,m} :=& Q^*_{22,mm} = 2\pi\prod_{k\in\mathbb{Z}}\frac{\sigma_{1,k}}{\sigma_{2,k}} + \ell^2_m, \quad \forall\, m\in\mathbb{Z}\\
K^*_{11,mr} =& Q^*_{11,mr} = \frac{\ell^2_m}{|m-r|} \quad \forall\, m,r\in\mathbb{Z},\ m\ne r\\
K^*_{22,mr} =& Q^*_{22,mr} = \frac{\ell^2_m}{|m-r|} + \frac{\ell^2_m}{\langle r\rangle} \quad \forall\, m,r\in\mathbb{Z},\ m\ne r\\
K^*_{12,mr} =& Q^*_{12,mr} = \frac{\ell^2_m}{\langle r\rangle} \quad \forall\, m,r\in\mathbb{Z}\\
K^*_{21,mr} =& Q^*_{21,mr} = \frac{\ell^2_m}{\langle r\rangle} \quad \forall\, m,r\in\mathbb{Z}.
\end{aligned}$$

We now show that $D^n\to D^*$ and $K^n\to K^*$ in operator norm. For any $h_1\in\ell^2$, taking into account that $D^n_{11,nn}=2$, one gets

$$\|(D^*_{11}-D^n_{11})h_1\|^2 = |(Q^*_{11,nn}-2)h_{1,n}|^2 + \sum_{m\ne n}\Big|\frac{1}{\mathrm{i}\pi}\int_{\Gamma_{1,m}}\Big(1-\frac{n\pi-m\pi}{\sigma_{1,n}-\lambda}\Big)\frac{\zeta_{1,m}(\lambda)}{w_{1,m}(\lambda)}\,\mathrm{d}\lambda\, h_{1,m}\Big|^2.$$

Note that $|(Q^*_{11,nn}-1)h_{1,n}|^2 = \|h_1\|\cdot\ell^1_n$. Moreover writing

$$1-\frac{n\pi-m\pi}{\sigma_{1,n}-\lambda} = \frac{\sigma_{1,n}-n\pi}{\sigma_{1,n}-\lambda} + \frac{\lambda-m\pi}{\sigma_{1,n}-\lambda}$$

and using Lemma 8.18, one sees that

$$\sum_{m\ne n}\Big|\int_{\Gamma_{1,m}}\frac{\sigma_{1,n}-n\pi}{\sigma_{1,n}-\lambda}\frac{\zeta_{1,m}(\lambda)}{w_{1,m}(\lambda)}\,\mathrm{d}\lambda\, h_{1,m}\Big|^2 = O(|\sigma_{1,n}-n\pi|^2\|h_1\|^2).$$

Using in addition that for $\lambda\in G_{1,m}$, $|\lambda-m\pi|\le|\lambda-\tau_{1,m}|+|\tau_{1,m}-m\pi|$ one gets

$$\sum_{m\ne n}\Big|\int_{\Gamma_{1,m}}\frac{\lambda-m\pi}{\sigma_{1,n}-\lambda}\frac{\zeta_{1,m}(\lambda)}{w_{1,m}(\lambda)}\,\mathrm{d}\lambda\, h_{1,m}\Big|^2 = O(\|h_1\|^2\big(\sup_{m\ne n}\frac{|\gamma_{1,m}|+|\tau_{1,m}-m\pi|^2}{|n-m|}\big)^2).$$

which can be bounded by

$$C\|h_1\|^2\Big(\sum_{|m-n|\le\frac{|n|}{2}}(|\gamma_{1,m}|^2+|\tau_{1,m}-m\pi|^2)+\frac{1}{n^2}\Big).$$

Since $\sigma_{1,n}-n\pi=\ell^2_n$, $\gamma_{1,m}=\ell^2_m$, and $\tau_{1,m}-m\pi=\ell^2_m$ we conclude that in operator norm $\lim_{n\to\infty}\|D^*_{11}-D^n_{11}\|=0$. Similarly, for $h_2\in\ell^2$, one gets

$$\|(D^*_{22}-D^n_{22})h_2\|^2 = \sum_{m\in\mathbb{Z}}\Big|\int_{\tilde{\Gamma}_{2,m}}\frac{\mu}{\sigma_{2,m}}\frac{\pi_m^2}{\mu^2}\big(1-\frac{\pi_n}{\sigma_{1,n}-\frac{1}{16\mu}}\big)\frac{\zeta_{2,m}(-\frac{1}{16\mu})}{w_{2,m}(-\frac{1}{16\mu})}\,\mathrm{d}\mu h_{2,m}\Big|^2.$$

Using again Lemma 8.18 and $\lim_{n\to\infty}(1-\frac{\pi_n}{\sigma_{1,n}})=0$ one sees that $\lim_{n\to\infty}\|D^*_{22}-D^n_{22}\|=0$. Next we turn to K^*_{11}. For $h_1\in\ell^2$,

$$\|(K^*_{11}-K^n_{11})h_1\|^2=\sum_{m\neq n}\big|\sum_{r\neq m}(Q^*_{11,mr}-Q^n_{11,mr})h_{1,r}\big|^2+\big|\sum_{r\neq n}(Q^*_{11,nr}-Q^n_{11,nr})h_{1,r}\big|^2.$$

Since by definition $Q^n_{11,nr}=0$ for any $r\neq n$ the Cauchy-Schwarz inequality yields

$$\big|\sum_{r\neq n}(Q^*_{11,nr}-Q^n_{11,nr})h_{1,r}\big|^2\leq\sum_{r\neq n}|Q^*_{11,nr}|^2\|h_1\|^2.$$

Since for $n\neq r$

$$Q^*_{11,nr}=\frac{\mathrm{i}}{\pi}\int_{\Gamma_{1,n}}\frac{\sigma_{1,n}-\lambda}{\sigma_{1,r}-\lambda}\frac{\zeta_{1,n}(\lambda)}{w_{1,n}(\lambda)}\,\mathrm{d}\lambda$$

one gets again by Lemma 8.18 that

$$\sum_{r\neq n}|Q^*_{11,nr}|^2=\Big(\sum_{r\neq n}\frac{1}{|n-r|^2}\Big)\cdot\ell^1_n.$$

Similarly, since by definition $Q^n_{11,mn}=0$ for $m\neq n$ one has

$$\sum_{m\neq n}\big|\sum_{r\neq m}(Q^*_{11,mr}-Q^n_{11,mr})h_{1,r}\big|^2\leq I+II$$

where

$$I:=\sum_{m\neq n}\big|\sum_{r\neq m,n}\frac{1}{\pi}\int_{\Gamma_{1,m}}\frac{\sigma_{1,m}-\lambda}{\sigma_{1,r}-\lambda}\Big(1-\frac{n\pi-m\pi}{\sigma_{1,n}-\lambda}\Big)\frac{\zeta_{1,m}(\lambda)}{w_{1,m}(\lambda)}\,\mathrm{d}\lambda\,h_{1,r}\big|^2$$

$$II:=\sum_{m\neq n}|Q^*_{11,mn}h_{1,n}|^2=\sum_{m\neq n}\Big|\frac{1}{\pi}\int_{\Gamma_{1,m}}\frac{\sigma_{1,m}-\lambda}{\sigma_{1,n}-\lambda}\frac{\zeta_{1,m}(\lambda)}{w_{1,m}(\lambda)}\,\mathrm{d}\lambda\,h_{1,n}\Big|^2.$$

Let us begin by estimating the latter sum:

$$II\leq C\sum_{m\neq n}\frac{|\gamma_{1,m}|^2}{|n-m|^2}\|h_1\|^2\leq C\sum_{|m|\geq|n|/2}|\gamma_{1,m}|^2\|h_1\|^2+C\frac{1}{n^2}\|\gamma_1\|^2\|h_1\|^2.$$

The sum I is estimated similarly:

$$\begin{aligned}I\leq&\sum_{m\neq n}\sum_{r\neq m,n}\Big(\sup_{\lambda\in G_{1,m}}\Big|\frac{\sigma_{1,m}-\lambda}{\sigma_{1,r}-\lambda}\Big|\frac{|n\pi-\sigma_{1,n}|+|\lambda-m\pi|}{|\sigma_{1,n}-\lambda|}|\zeta_{1,m}(\lambda)|\Big)^2\|h_1\|^2\\ \leq&C\|h_1\|^2\sum_{m\neq n}|\gamma_{1,m}|^2\frac{|n\pi-\sigma_{1,n}|^2+|\tau_{1,m}-m\pi|^2+|\gamma_{1,m}|^2}{|n-m|^2}\sum_{r\neq m,n}\frac{1}{|r-m|^2}\end{aligned}$$

leading to an estimate of the same kind as for the sum II. As a result we conclude that in the ℓ^2-operator norm

$$\lim_{n\to\infty}\|K^*_{11}-K^n_{11}\|=0.$$

In the same way one shows that

$$\lim_{n\to\infty}\|K^*_{22}-K^n_{22}\|=0,\quad\lim_{n\to\infty}\|K^*_{12}-K^n_{12}\|=0,\quad\text{and}\quad\lim_{n\to\infty}\|K^*_{21}-K^n_{21}\|=0.$$

Hence we have established that

$$\lim_{n\to\infty}\|D^*-D^n\|=0,\quad\lim_{n\to\infty}\|K^*-K^n\|=0.$$

Going trough the arguments of the proof one verifies that the convergence is locally uniform in $\Omega_{r,*}$. For any $n\geq 0$, Q^n is a continuous map $Q^n:\Omega_{r,*}\to\mathcal{L}(\ell^2\times\ell^2)$. By the locally uniform convergence $Q^n\to Q^*$ for $n\to\infty$ it then follows that $Q^*:\Omega_{r,*}\to\mathcal{L}(\ell^2\times\ell^2)$ is continuous as well. Arguing as in the proof of Lemma 8.21, Q^* is boundedly invertible at every point of $\Omega_{r,*}$. Since or any given $v\in H^1_r$, the set

$$\Pi(v):=\{\ s=(s_1,s_2)\in\ell^2_{\mathbb{R}}\times\ell^2_{\mathbb{R}}\ :\ \sigma_{1,k}\in G_{1,k}(v),\ (-16\sigma_{2,k})^{-1}\in G_{2,k}(v)\ \ \forall\ k\in\mathbb{Z}\ \}$$

is compact in $\ell^2_{\mathbb{R}}\times\ell^2_{\mathbb{R}}$, the operator $Q^*(v,s)$ is indeed uniformly boundedly invertible for any $s\in\Pi(v)$. By continuity, $Q^n(v,s)$ is also uniformly boundedly invertible for all large n and $s\in\Pi(v)$, and hence by Corollary 8.22, for all $n\geq 0$. This finishes the proof of Lemma 8.25. □

Summarizing our results so far, the functions $\psi_n(\lambda) = -\frac{1}{\pi_n}\frac{1}{\psi_{n,2}(\infty)}\psi_{n,1}(\lambda)\psi_{n,2}(\lambda)$ with

$$\psi_{n,1}(\lambda) = \prod_{k\neq n}\frac{\sigma^n_{1,k}-\lambda}{\pi_k}, \quad \psi_{n,2}(\lambda) = \prod_{k\in\mathbb{Z}}\frac{\sigma^n_{2,k}+\frac{1}{16\lambda}}{\pi_k}$$

satisfy

$$\int_{\Gamma_{1,m}}\frac{\psi_n(\lambda)}{\sqrt[c]{\chi_p(\lambda)}}\,\mathrm{d}\lambda = 0 \ \ \forall\, m\neq n, \quad \int_{\Gamma_{2,m}}\frac{\psi_n(\lambda)}{\sqrt[c]{\chi_p(\lambda)}}\,\mathrm{d}\lambda = 0 \ \ \forall\, m\in\mathbb{Z}. \tag{8.54}$$

We now check that they also satisfy the normalization condition

$$\frac{1}{2\pi}\int_{\Gamma_{1,n}}\frac{\psi_n(\lambda)}{\sqrt[c]{\chi_p(\lambda)}}\,\mathrm{d}\lambda = 1.$$

***Lemma* 8.26** *On the complex neighborhood $W\subset\hat{W}$ of H^1_r of Lemma 8.24 one has for any $n\geq 0$*

$$\int_{\Gamma_{1,n}}\frac{\psi_n(\lambda)}{\sqrt[c]{\chi_p(\lambda)}}\,\mathrm{d}\lambda = 2\pi.$$

Proof. Let $v\in W$ and $n\geq 0$ be arbitrary. By (8.54) it follows by Cauchy's theorem that for $N\geq 1$ sufficiently large so that $U_{j,m}\cap\partial B_N=\emptyset$, $U_{j,m}\cap\partial B_{-N}=\emptyset$ $\forall\, m\in\mathbb{Z}$, $k=1,2$, one has

$$\int_{\Gamma_{1,n}}\frac{\psi_n(\lambda)}{\sqrt[c]{\chi_p(\lambda)}}\,\mathrm{d}\lambda = I_N - II_N \tag{8.55}$$

where

$$I_N := \int_{\partial B_N}\frac{\psi_n(\lambda)}{\sqrt[c]{\chi_p(\lambda)}}\,\mathrm{d}\lambda, \quad II_N := \int_{\partial B_{-N}}\frac{\psi_n(\lambda)}{\sqrt[c]{\chi_p(\lambda)}}\,\mathrm{d}\lambda.$$

Let us first compute I_N. Using that $\sqrt[c]{\chi_1(0)} = \sqrt[c]{\chi_2(\infty)}$, one has $\sqrt[c]{\chi_p(\lambda)} = \mathrm{i}\sqrt[c]{\chi_1(\lambda)}\sqrt[c]{\chi_2(\lambda)}/\sqrt[c]{\chi_2(\infty)}$. Letting $\sigma^n_{1,n}:=\tau_{1,n}$, the contour integral I_N can be written as

$$I_N = \frac{1}{\mathrm{i}}\int_{\partial B_N}\frac{1}{\lambda-\sigma^n_{1,n}}\Big(\prod_{k\in\mathbb{Z}}\frac{\sigma_{1,k}-\lambda}{w_{1,k}(\lambda)}\Big)\frac{\psi_{n,2}(\lambda)/\psi_{n,2}(\infty)}{\sqrt[c]{\chi_2(\lambda)}/\sqrt[c]{\chi_2(\infty)}}\,\mathrm{d}\lambda.$$

Note that as $N\to\infty$

$$\sup_{\lambda\in\partial B_N}|\psi_{n,2}(\lambda)/\psi_{n,2}(\infty)-1| = O(\frac{1}{N}), \quad \sup_{\lambda\in\partial B_N}|\sqrt[c]{\chi_2(\lambda)}/\sqrt[c]{\chi_2(\infty)}-1| = O(\frac{1}{N}).$$

Furthermore, by Lemma B.5

$$\sup_{\lambda\in\partial B_N}\Big|\prod_{k\in\mathbb{Z}}\frac{\sigma^n_{1,k}-\lambda}{w_{1,k}(\lambda)}-1\Big| = o(1) \quad \text{as } N\to\infty.$$

Hence we get by Cauchy's theorem

$$\lim_{N\to\infty} I_N = \lim_{N\to\infty}\frac{1}{\mathrm{i}}\int_{\partial B_N}\frac{1}{\lambda-\sigma^n_{1,n}}(1+o(1))\,\mathrm{d}\lambda = 2\pi. \tag{8.56}$$

Now let us turn towards II_N. By the change of variable of integration $\lambda = -\frac{1}{16\mu}$,

$$II_N = \int_{\partial B_{-N}}\frac{\psi_n(\lambda)}{\sqrt[c]{\chi_p(\lambda)}}\,\mathrm{d}\lambda = \int_{\partial B_N}\frac{\psi_{n,1}(-\frac{1}{16\mu})}{\sqrt[c]{\chi_p(-\frac{1}{16\mu})}}\frac{\mathrm{d}\mu}{16\mu^2} = \int_{\partial B_N}\frac{\mathrm{i}}{\pi_n}\frac{\psi_{n,1}(-\frac{1}{16\mu})}{\sqrt[c]{\chi_1(-\frac{1}{16\mu})}/\sqrt[c]{\chi_1(0)}}\prod_{k\in\mathbb{Z}}\frac{\sigma^n_{2,k}-\mu}{w_{2,k}(-\frac{1}{16\mu})}\frac{\mathrm{d}\mu}{16\mu^2}.$$

By the definition of $w_{2,k}$, one has $w_{2,k}(-\frac{1}{16\mu}) = \sqrt[s]{(\lambda^+_{2,k}-\mu)(\lambda^-_{2,k}-\mu)}$. Hence again by Lemma B.5 one has

$$\sup_{\mu\in\partial B_N}\Big|\prod_{k\in\mathbb{Z}}\frac{\sigma^n_{2,k}-\mu}{w_{2,k}(-\frac{1}{16\mu})}-1\Big| = o(1) \quad \text{as } N\to\infty.$$

Furthermore, as $N \to \infty$

$$\sup_{\mu \in \partial B_N} |\psi_{n,1}(-\frac{1}{16\mu}) - \psi_{n,1}(0)| = O(\frac{1}{N}), \quad \sup_{\mu \in \partial B_N} |\frac{\sqrt[c]{\chi_1(-\frac{1}{16\mu})}}{\sqrt[c]{\chi_1(0)}} - 1| = O(\frac{1}{N}).$$

Altogether, one then concludes that

$$\lim_{N\to\infty} II_N = \lim_{N\to\infty} \int_{\partial B_N} \frac{\mathrm{i}}{\pi_n} \psi_{n,1}(0)(1+o(1)) \frac{\mathrm{d}\mu}{16\mu^2} = 0. \tag{8.57}$$

Combining (8.56) - (8.57) with (8.55) yields the claimed identity. □

To finish the proof of Theorem 8.12 it remains to establish the claimed estimates of $\sigma^n_{1,m}$, $\sigma^n_{2,m}$. To finish the proof of Theorem 8.12 it remains to establish the claimed estimates of $\sigma^n_{1,m}$, $\sigma^n_{2,m}$. Recall that for convenience we set $\sigma^n_{1,n} = \tau_{1,n}$.

***Lemma* 8.27** *On the complex neighborhood $W \subset \hat{W}$ of H^1_r of Lemma 8.24 one has for $j = 1, 2$*

$$\sigma^n_{j,m} = \tau_{j,m} + \gamma^2_{j,m} \ell^2_m \tag{8.58}$$

uniformly in $n \geq 0$ and locally uniformly on W. It means that $|\sigma^n_{j,m} - \tau_{j,m}| \leq |\gamma_{j,m}|^2 a^n_{j,m}$ where $a^n_{j,m} \geq 0$ and $\sum_m (|a^n_{1,m}|^2 + |a^n_{2,m}|^2) \leq C$. The constant C can be chosen uniformly in n and locally uniformly on W.

Proof. For $v \in W$, let $(U_{j,m})_{m\in\mathbb{Z}}$ be the neighborhoods, defined in terms of isolating neighborhoods U_m, $m \in \mathbb{Z}$, at the beginning of this section. Let $n \geq 0$ be arbitrary. First we treat the case $j = 1$. Multiplying the identity, $F^n_{1,m}(v, s^n(v)) = 0$ by π it can be written as

$$0 = \int_{\Gamma_{1,m}} \frac{\sigma^n_{1,m} - \lambda}{w_{1,m}(\lambda)} \chi^n_{1,m}(\lambda)\, \mathrm{d}\lambda \tag{8.59}$$

where $\chi^n_{1,m}(\lambda) = \frac{\pi(n-m)}{\sigma^n_{1,n} - \lambda} \zeta^n_{1,m}(\lambda)$ with $\sigma^n_{1,n} = \tau_{1,n}$ and (cf (8.44) - (8.45))

$$\zeta^n_{1,m} := \mathrm{i}\Big(\prod_{k\neq m} \frac{\sigma^n_{1,k} - \lambda}{w_{1,k}(\lambda)}\Big) \frac{\psi_{n,2}(\lambda)/\psi_{n,2}(\infty)}{\sqrt[c]{\chi_2(\lambda)}/\sqrt[c]{\chi_2(\infty)}}.$$

Expanding $\chi^n_{1,m}$ at $\lambda = \tau_{1,m}$ up to first order, one has

$$\chi^n_{1,m}(\lambda) = \chi^n_{1,m}(\tau_{1,m}) + (\lambda - \tau_{1,m}) b^n_{1,m}(\lambda).$$

Since $\int_{\Gamma_{1,m}} \frac{\tau_{1,m} - \lambda}{w_{1,m}(\lambda)}\, \mathrm{d}\lambda = 0$ and $\int_{\Gamma_{1,m}} \frac{1}{w_{1,m}(\lambda)}\, \mathrm{d}\lambda = -2\pi\mathrm{i}$ one has

$$\frac{1}{2\pi} \int_{\Gamma_{1,m}} \frac{\sigma^n_{1,m} - \lambda}{w_{1,m}(\lambda)} \chi^n_{1,m}(\tau_{1,m})\, \mathrm{d}\lambda = -(\sigma^n_{1,m} - \tau_{1,m}) \chi^n_{1,m}(\tau_{1,m})$$

and hence (8.59) becomes

$$\chi^n_{1,m}(\tau_{1,m})(\sigma^n_{1,m} - \tau_{1,m}) = \frac{1}{2\pi\mathrm{i}} \int_{\Gamma_{1,m}} \frac{\sigma^n_{1,m} - \lambda}{w_{1,m}(\lambda)} (\lambda - \tau_{1,m}) b^n_{1,m}(\lambda)\, \mathrm{d}\lambda. \tag{8.60}$$

Note that by (8.46), $\sup_{\lambda \in U_{1,m}} |\zeta_{1,m}(\lambda) - 1| = \ell^2_m$ and by the definition of $U_{1,n}$, $\sigma^n_{1,n} = \tau_{1,n} \in U_{1,n}$, implying that $\sup_{\lambda \in U_{1,m}} \big|\frac{\pi(n-m)}{\sigma^n_{1,n} - \lambda} - 1\big| = \ell^2_m$. Hence by the definition of $\chi^n_{1,m}(\lambda)$

$$\sup_{\lambda \in U_{1,m}} \big|\chi^n_{1,m}(\lambda) - \mathrm{i}\big| = \ell^2_m, \quad \inf_{m\neq n} |\chi^n_{1,m}(\lambda)| > 0.$$

By choosing $\Gamma_{1,m}$ so that $\inf_m \operatorname{dist}(\Gamma_{1,m}, \partial U_{1,m}) > 0$ one then gets by Cauchy's estimate that

$$\sum_{m\neq n} \sup_{\lambda \in \Gamma_{1,m}} |b^n_{1,m}(\lambda)|^2 \leq C$$

for some $C > 0$. It then follows from Lemma 8.18 applied to (8.60) that for $m \neq n$

$$|\sigma^n_{1,m} - \tau_{1,m}| = \max_{\lambda \in G_{1,m}} |\sigma^n_{1,m} - \lambda||\gamma_{1,m}|\ell^2_m.$$

Since for any $\lambda \in G_{1,m}$, $|\sigma^n_{1,m} - \lambda| \le |\sigma^n_{1,m} - \tau_{1,m}| + |\gamma_{1,m}/2|$ one gets

$$|\sigma^n_{1,m} - \tau_{1,m}| = (|\sigma^n_{1,m} - \tau_{1,m}| + |\gamma_{1,m}/2|)|\gamma_{1,m}|\ell^2_m \tag{8.61}$$

implying that $|\sigma^n_{1,m} - \tau_{1,m}| = |\gamma_{1,m}|\ell^2_m$. Substituting the latter estimate into the right hand side of (8.61) one obtains $|\sigma^n_{1,m} - \tau_{1,m}| = |\gamma_{1,m}|^2 a^n_{1,m}$ for some $a_{1,m} \ge 0$ (and $a^n_{1,n} = 0$) with $\sum_m |a^n_{1,m}|^2 \le C$. Going trough the arguments of the proof one verifies that $C > 0$ can be chosen uniformly in $n \ge 0$ and locally uniformly on $V_v \subset W$. The case $j = 2$ is treated in a similar fashion. □

8.3 Angles

In this section we introduce the angle coordinates $\theta_n(v)$ for $n \ge 0$ on the open set $W \setminus Z_n$ where W is the complex neighborhood of H^1_r of Theorem 8.12 on which the ψ-functions ψ_n, $n \ge 0$ are defined and Z_n denotes the subvariety of $\hat{W}$ given by $Z_n = \{ v \in \hat{W} : \lambda^-_n \neq \lambda^+_n \}$. The angle coordinate $\theta_{-n}(q,p)$, for $n \ge 1$ will be defined in terms of $\theta_n(-q,p)$ at the end of this section. Without further reference we will use the notation introduced in the previous sections, in particular the one introduced in Section 6.3, 6.4, and 8.2. In addition we introduce for any $m \in \mathbb{Z}$ the sets

$$\tilde{U}_{1,m} := U_{1,m}, \quad \tilde{U}_{2,m} := \{ -\frac{1}{16\lambda} : \lambda \in U_{2,m} \}.$$

For any $n \ge 0$ and $v \in W \setminus Z_n$ we set

$$\theta_n(v) := \eta_n(v) + \sum_{m \neq n} \beta^n_{1,m}(v) + \sum_{m \in \mathbb{Z}} \beta^n_{2,m}(v) \tag{8.62}$$

where

$$\eta_n(v) \equiv \beta^n_{1,n} := \int_{\lambda^-_{1,n}}^{\mu_{1,n}} \frac{\psi_n(\lambda)}{\sqrt[*]{\chi_p(\lambda)}} \mathrm{d}\lambda \pmod{2\pi} \tag{8.63}$$

$$\beta^n_{1,m}(v) := \int_{\lambda^-_{1,m}}^{\mu_{1,m}} \frac{\psi_n(\lambda)}{\sqrt[*]{\chi_p(\lambda)}} \mathrm{d}\lambda, \quad \forall\, m \neq n, \tag{8.64}$$

and

$$\beta^n_{2,m}(v) := \int_{\lambda^-_{2,m}}^{\mu_{2,m}} \frac{\psi_n(-\frac{1}{16\mu})}{\sqrt[*]{\chi_p(-\frac{1}{16\mu})}} \frac{\mathrm{d}\mu}{16\mu^2}, \quad \forall\, m \in \mathbb{Z}. \tag{8.65}$$

The integrals in the definition of $\beta^n_{j,m}$ are taken along any path from $\lambda^-_{j,m}$ to $\mu_{j,m}$ inside the neighborhood $\tilde{U}_{j,m}$ that besides its starting point $\lambda^-_{j,m}$ and maybe its end point $\mu_{j,m}$ contains no point of $[\lambda^-_{j,m}, \lambda^+_{j,m}]$. For $\mu_{j,m} \neq \lambda^\pm_{j,m}$, the sign of the $*$-root $\sqrt[*]{\Delta^2(\lambda) - 1}$ along such a path in $\tilde{U}_{j,m}$ is chosen so that

$$\sqrt[*]{\Delta^2(\mu_m) - 1} = \delta(\mu_m). \tag{8.66}$$

In the case where $\mu_{j,m} \in \{\lambda^-_{j,m}, \lambda^+_{j,m}\}$ the choice of the sign does not matter since for any choice of the sign

$$\int_{\lambda^-_{1,m}}^{\lambda^+_{1,m}} \frac{\psi_n(\lambda)}{\sqrt{\chi_p(\lambda)}} \mathrm{d}\lambda = 0 \ (m \neq n), \quad \int_{\lambda^-_{2,m}}^{\lambda^+_{2,m}} \frac{\psi_n(-\frac{1}{16\mu})}{\sqrt{\chi_p(-\frac{1}{16\mu})}} \frac{\mathrm{d}\mu}{16\mu^2} = 0 \ (m \in \mathbb{Z}) \tag{8.67}$$

and

$$\int_{\lambda^-_{1,n}}^{\lambda^+_{1,n}} \frac{\psi_n(\lambda)}{\sqrt{\chi_p(\lambda)}} \mathrm{d}\lambda = \pi \pmod{2\pi}. \tag{8.68}$$

Note that the integrals in (8.63)-(8.65) are improper integrals. They exist since whenever $\lambda^-_{j,m} \neq \lambda^+_{j,m}$ the integrand $\psi_n(\lambda)/\sqrt[*]{\chi_p(\lambda)}$ is of the order $(\lambda - \lambda^\pm_{j,m})^{-1/2}$ near $\lambda^\pm_{j,m}$. On the other hand, if $\lambda^-_{j,m} = \lambda^+_{j,m}$, then by Theorem 8.12, $\tau_{j,m}$ is a root of ψ_n and by the definition of the canonical root $\sqrt[c]{\chi_p(\lambda)}$ (which equals $\sqrt[*]{\chi_p(\lambda)}$ up to a sign) has a factor $(\tau_{j,m}-\lambda)$ implying that $\psi_n(\lambda)/\sqrt[*]{\chi_p(\lambda)}$ is analytic on $\tilde{U}_{j,m}$. Furthermore, the integrals in (8.63) - (8.65) are independent of the chosen pathes since in view of Theorem 8.12 the integrals along any closed loop in $\tilde{U}_{j,m}$ around $[\lambda^-_{j,m},\lambda^+_{j,m}]$ vanishes for $(j,m)\neq(1,n)$ while such a loop around $[\lambda^-_{1,n},\lambda^+_{1,n}]$ contributes a multiple of 2π and hence does not change η_n since it is defined mod 2π. We also note that by the change of variable $\lambda\mapsto\mu:=-\lambda$ one gets for any $m\geq 1$,

$$\beta^n_{1,-m}=\int_{\lambda^-_{1,-m}}^{\mu_{1,-m}}\frac{\psi_n(\lambda)}{\sqrt[*]{\chi_p(\lambda)}}\,\mathrm{d}\lambda=-\int_{\lambda^-_{1,m}}^{\mu_{1,m}}\frac{\psi_n(-\mu)}{\sqrt[*]{\chi_p(\mu)}}\,\mathrm{d}\mu \tag{8.69}$$

where we have used that by Lemma 2.14,

$$\delta(-\lambda)=\delta(\lambda),\ \Delta(-\lambda)=\Delta(\lambda),\quad \text{and hence}\quad \sqrt[*]{\chi_p(-\lambda)}=\sqrt[*]{\chi_p(\lambda)}.$$

Similarly for any $m\geq 1$

$$\beta^n_{2,-m}=\int_{\lambda^-_{2,-m}}^{\mu_{2,-m}}\frac{\psi_n(-\frac{1}{16\mu})}{\sqrt[*]{\chi_p(-\frac{1}{16\mu})}}\frac{\mathrm{d}\mu}{16\mu^2}=-\int_{\lambda^-_{2,m}}^{\mu_{2,m}}\frac{\psi_n(\frac{1}{16\mu})}{\sqrt[*]{\chi_p(-\frac{1}{16\mu})}}\frac{\mathrm{d}\mu}{16\mu^2}. \tag{8.70}$$

Finally we note that actually $\beta^n_{1,m}$ $(m\neq n)$ and $\beta^n_{2,m}$ $(m\in\mathbb{Z})$ are well defined on all of W and real valued on H^1_r, whereas η_n is real valued on $H^1_r\setminus Z_n$.

In a first step we need to estimate the functions $\beta^n_{j,m}$ on $W\setminus Z_n$ to confirm that the infinite sums in (8.62) converge and hence $\theta_n(v)$ is well defined on $W\setminus Z_n$. Let

$$d_{j,m}:=\min\{|\mu_{j,m}-\lambda^-_{j,m}|,|\mu_{j,m}-\lambda^+_{j,m}|\}$$

***Lemma* 8.28** *For any $n\geq 0$, the following estimates holds locally uniformly on W:*

$$\beta^n_{1,m}=O\left(\frac{\sqrt[+]{|\gamma_{1,m}|+d_{1,m}}\cdot\sqrt[+]{d_{1,m}}}{m-n}\right),\qquad\forall\ m\neq n \tag{8.71}$$

$$\beta^n_{2,m}=O\left(\frac{\sqrt[+]{|\gamma_{2,m}|+d_{2,m}}\cdot\sqrt[+]{d_{2,m}}}{\pi_n m^2}\right),\qquad\forall\ m\in\mathbb{Z} \tag{8.72}$$

and as a consequence

$$\beta^n_{1,m}=O\left(\frac{|\gamma_{1,m}|+|\mu_{1,m}-\tau_{1,m}|}{m-n}\right),\qquad\forall\ m\neq n \tag{8.73}$$

$$\beta^n_{2,m}=O\left(\frac{|\gamma_{2,m}|+|\mu_{2,m}-\tau_{2,m}|}{\pi_n m^2}\right),\qquad\forall\ m\in\mathbb{Z} \tag{8.74}$$

Proof. We argue as in the proof of Lemma 8.17 in Section 8.2. First we prove (8.71). Let $m\neq n$ and $v_0\in W$. Furthermore, let V_{v_0} be a neighborhood of v_0 in W, and U_m, $m\in\mathbb{Z}$ isolating neighborhoods which work for any v in V_{v_0} (cf Section 6.2). By (8.67), $\beta^n_{1,m}=0$ if $\mu_{1,m}\in\{\lambda^-_{1,m},\lambda^+_{1,m}\}$. Thus it remains to consider the case $\mu_{1,m}\neq\lambda^\pm_{1,m}$. By (8.67), $\beta^n_{1,m}$ does not change if we interchange the role of $\lambda^-_{1,m}$ and $\lambda^+_{1,m}\zeta$ and we assume that v_0 is in

$$W^+_{1,m}:=\{\ v\in W\ :\ |\mu_{1,m}-\lambda^-_{1,m}|\leq|\mu_{1,m}-\lambda^+_{1,m}|\ \}.$$

Note that $W=W^+_{1,m}\cup W^-_{1,m}$, where W^-_m is defined as $W^+_{1,m}$, but with the roles of $\lambda^-_{1,m}$ and $\lambda^+_{1,m}$ interchanged. For $\lambda\in U_{1,m}$ we write similarly as in (8.35)

$$\frac{\psi_n(\lambda)}{\sqrt[c]{\chi_p(\lambda)}}=\frac{\sigma^n_{1,m}-\lambda}{w_{1,m}(\lambda)}\zeta^n_{1,m}(\lambda) \tag{8.75}$$

with

$$\zeta^n_{1,m}(\lambda)=\frac{\mathrm{i}}{w_{1,n}(\lambda)}\Big(\prod_{k\neq m,n}\frac{\sigma^n_k-\lambda}{w_{1,k}(\lambda)}\Big)\frac{\psi_{n,2}(\lambda)/\psi_{n,2}(\infty)}{\sqrt[c]{\chi_2(\lambda)}/\sqrt[c]{\chi_2(\infty)}}. \tag{8.76}$$

Note that $\zeta^n_{1,m}(\lambda)$ is analytic on $U_{1,m}\times V_{v_0}$ and by (8.36) satisfies

$$\zeta^n_{1,m}(\lambda)=O(\frac{1}{|m-n|}). \tag{8.77}$$

To estimate $\beta^n_{1,m} = \beta^n_{1,m}(v_0)$ let $\omega_{1,m} := \mu_{1,m} - \lambda^-_{1,m}$. By assumption $\omega_{1,m} \neq 0$. The substitution $\lambda(t) := \lambda^-_{1,m} + t\omega_{1,m}$ then leads to

$$\int_{\lambda^-_{1,m}}^{\mu_{1,m}} \frac{\sigma^n_{1,m} - \lambda}{w_{1,m}(\lambda)} \zeta^n_{1,m}(\lambda)\,\mathrm{d}\lambda = O\Big(\frac{1}{|n-m|} \int_0^1 \big|\frac{\sigma^n_{1,m} - \lambda(t)}{t\omega_{1,m}}\big|^{1/2} \big|\frac{\sigma^n_{1,m} - \lambda(t)}{\lambda^+_{1,m} - \lambda(t)}\big|^{1/2} |\omega_{1,m}|\,\mathrm{d}t\Big).$$

Since by assumptions $|\mu_{1,m} - \lambda^-_{1,m}| \le |\mu_{1,m} - \lambda^+_{1,m}|$ one has $|\lambda^+_{1,m} - \lambda(t)| \ge |\gamma_{1,m}|/2$ for any $0 \le t \le 1$, yielding the estimate

$$\big|\frac{\sigma^n_{1,m} - \lambda(t)}{\lambda^+_{1,m} - \lambda(t)}\big| = \big|1 + \frac{\sigma^n_{1,m} - \lambda^+_{1,m}}{\lambda^+_{1,m} - \lambda(t)}\big| = O(1). \tag{8.78}$$

On the other hand

$$\big|\frac{\sigma^n_{1,m} - \lambda(t)}{t\omega_{1,m}}\big|^{1/2} \le \frac{(|\sigma^n_{1,m} - \lambda^-_{1,m}| + |\omega_{1,m}|)^{1/2}}{\sqrt{t}|\omega_{1,m}|^{1/2}} = O\left(\frac{(|\gamma_{1,m}| + |\omega_{1,m}|)^{1/2}}{\sqrt{t}|\omega_{1,m}|^{1/2}}\right). \tag{8.79}$$

It then follows that

$$\Big|\int_{\lambda^-_{1,m}}^{\mu_{1,m}} \frac{\sigma^n_{1,m} - \lambda}{w_{1,m}(\lambda)} \zeta^n_{1,m}(\lambda)\,\mathrm{d}\lambda\Big| = O\left(\frac{(|\gamma_{1,m}| + |\omega_{1,m}|)^{1/2}|\omega_{1,m}|^{1/2}}{|n-m|}\right).$$

Going trough the arguments of the proof one sees that the claimed estimates (8.71) hold locally uniformly on W. The estimates (8.72) are obtained in a similar way. Let $m \in \mathbb{Z}$ and $v_0 \in W$. By (8.67), $\beta^n_{2,m} = 0$ if $\mu_{2,m} \in \{\lambda^-_{2,m}, \lambda^+_{2,m}\}$. Thus again it remains to consider the case $\mu_{2,m} \neq \lambda^\pm_{2,m}$. By (8.67), $\beta^n_{2,m}$ does not change if we interchange the role of $\lambda^-_{2,m}$ and $\lambda^+_{2,m}$, and we assume that v_0 is in

$$W^+_{2,m} := \{\, v \in W \,:\; |\mu_{2,m} - \lambda^-_{2,m}| \le |\mu_{2,m} - \lambda^+_{2,m}| \,\}.$$

Note that $W = W^+_{2,m} \cup W^-_{2,m}$, where $W^-_{2,m}$ is defined as $W^+_{2,m}$, but with the roles of $\lambda^-_{2,m}$ and $\lambda^+_{2,m}$ interchanged. For $\lambda \in \tilde{U}_{2,m}$ $(= \{\, -\frac{1}{16\lambda} \,:\; \lambda \in U_{2,m} \,\})$ we write similarly as in (8.37)

$$\frac{\psi_n(-\frac{1}{16\mu^2})}{\sqrt[c]{\chi_p(-\frac{1}{16\mu^2})}} = \frac{1}{\pi_n} \frac{\sigma^n_{2,m}\mu}{w_{2,m}(-\frac{1}{16\mu})} \zeta^n_{2,m}(-\frac{1}{16\mu}) \tag{8.80}$$

where $w_{2,m}(-\frac{1}{16\mu}) = \sqrt[s]{(\lambda^+_{2,m} - \mu)(\lambda^-_{2,m} - \mu)}$ and

$$\zeta^n_{2,m}(-\frac{1}{16\mu}) = \frac{\mathrm{i}\psi_{n,1}(-\frac{1}{16\mu})}{\sqrt[c]{\chi_1(-\frac{1}{16\mu})/\sqrt[c]{\chi_1(0)}}} \frac{1}{\psi_{n,2}(\infty)} \Big(\prod_{k \neq m} \frac{\sigma^n_{2,k} - \mu}{w_{2,k}(-\frac{1}{16\mu})}\Big) \tag{8.81}$$

Note that $\zeta^n_{2,m}(-\frac{1}{16\mu})$ as a function of μ is analytic on $\tilde{U}_{2,m} \times V_{v_0}$ and by (8.39) satisfies

$$\zeta^n_{2,m}(-\frac{1}{16\mu}) = O(1).$$

To estimate $\beta^n_{2,m} = \beta^n_{2,m}(v_0)$ let $\omega_{2,m} := \mu_{2,m} - \lambda^-_{2,m}$. By assumption $\omega_{2,m} \neq 0$. The substitution $\mu(t) := \lambda^-_{2,m} + t\omega_{2,m}$ then leads to

$$\int_{\lambda^-_{2,m}}^{\mu_{2,m}} \frac{\sigma^n_{2,m} - \mu}{\sqrt[s]{(\lambda^+_{2,m} - \mu)(\lambda^-_{2,m} - \mu)}} \zeta^n_{2,m}(-\frac{1}{16\mu}) \frac{\mathrm{d}\mu}{16\mu^2} = O\Big(\frac{1}{m^2} \int_0^1 \big|\frac{\sigma^n_{2,m} - \mu(t)}{t\omega_{2,m}}\big|^{1/2} \big|\frac{\sigma^n_{2,m} - \mu(t)}{\lambda^+_{2,m} - \mu(t)}\big|^{1/2} |\omega_{2,m}|\,\mathrm{d}t\Big).$$

Arguing as in the case $j = 1$ one obtains

$$\Big|\int_{\lambda^-_{2,m}}^{\mu_{2,m}} \frac{\sigma^n_{2,m} - \mu}{w_{2,m}(-\frac{1}{16\mu})} \zeta^n_{1,m}(-\frac{1}{16\mu}) \frac{\mathrm{d}\mu}{16\mu^2}\Big| = O\left(\frac{(|\gamma_{2,m}| + |\omega_{2,m}|)^{1/2}|\omega_{2,m}|^{1/2}}{m^2}\right).$$

Going trough the arguments of the proof one sees that the claimed estimates (8.72) hold locally uniformly on W. □

***Lemma* 8.29** *Let $n \geq 0$. For any $(j,m) \neq (1,n)$, the functions $\beta^n_{j,m}$ are real analytic on W while the function η_n is real analytic on $W \setminus Z_n$ if taken modulo π.*

Remark 8.30. The values of η_n have to be taken modulo π as, due to the ordering and to possible crossings, the periodic eigenvalues $\lambda^-_{1,n}$, $\lambda^+_{1,n}$ might not be continuous on W.

Proof. It is convenient to introduce the subsets

$$Z_{j,m} := \{ \, v \in W \, : \, \gamma^2_{j,m} = 0 \, \}, \quad E_{j,m} := \{ \, v \in W \, : \, \Delta(\mu_{j,m}) = 2(-1)^m \, \}.$$

Since $\gamma^2_{j,m}$ and $\mu_{j,m}$ are analytic functions on W (and Δ is analytic on $\mathbb{C}^* \times H^1_c$), these subsets are analytic subvarieties of W. We are going to prove that for any $(j,m) \neq (1,n)$, $\beta^n_{j,m}$ is analytic on $W \setminus (Z_{j,m} \cup E_{j,m})$, extends continuously to W, and has weakly analytic restrictions to $Z_{j,m}$ and $E_{j,m}$. It then follows from Theorem A.6 that $\beta^n_{j,m}$ is analytic on W. First let us consider the case $j = 1$. To prove that $\beta^n_{1,m}$, $m \neq n$, is analytic on $W \setminus Z_{1,m} \cup E_{1,m}$ it is to prove that it is complex differentiable at any potential $v_0 \in W \setminus Z_{1,m} \cup E_{1,m}$. Note that although $\lambda^\pm_{1,m}$ are simple eigenvalues outside $Z_{1,m}$, they are not necessarily continuous but by Lemma 5.13, there exist two analytic functions $\rho^+_{1,m}$, $\rho^-_{1,m}$, defined on some neighborhood of v_0 continued in $W \setminus (Z_{1,m} \cup E_{1,m})$ such that the set equality $\{\lambda^-_{1,m}, \lambda^+_{1,m}\} = \{\rho^-_{1,m}, \rho^+_{1,m}\}$ holds. Chose $\rho^-_{1,m}$ so that near v_0,

$$|t\mu_{1,m} + (1-t)\rho^-_{1,m} - \rho^+_{1,m}| \geq \frac{1}{3}|\gamma_{1,m}| \quad \forall \, 0 \leq t \leq 1.$$

In view of (8.67), we thus can write

$$\beta^n_{1,m} = \int_{\rho^-_{1,m}}^{\mu_{1,m}} \frac{\psi_n(\lambda)}{\sqrt[*]{\chi_p(\lambda)}} \, \mathrm{d}\lambda.$$

Similarly as in (8.75) - (8.76), write

$$\frac{\psi_n(\lambda)}{\sqrt[c]{\chi_p(\lambda)}} = \frac{\sigma^n_{1,m} - \lambda}{w_{1,m}(\lambda)} \zeta^n_{1,m}(\lambda)$$

and let $\omega_{1,m} := \mu_{1,m} - \rho^-_{1,m}$. For $\lambda \equiv \lambda(t) = \rho^-_{1,m} + t\omega_{1,m}$ one has $w_{1,m}(\lambda)^2 = t\omega_{1,m} \cdot (\lambda - \rho^+_{1,m})$ and since by assumption, $|\lambda - \rho^+_{1,m}| \geq |\gamma_{1,m}/3|$ for $0 \leq t \leq 1$ and v near v_0, $\{ \, \arg(\lambda(t) - \rho^+_{1,m}) \, : \, 0 \leq t \leq 1 \, \}$ is contained in an interval of length strictly smaller than π. Hence the square root $\sqrt{\lambda(t) - \rho^+_{1,m}}$ can be chosen to be continuous in t and analytic in v for v near v_0. With the appropriate choice of the root $\sqrt{\omega_{1,m}}$ it then follows that

$$\beta^n_{1,m} = \int_0^1 \frac{1}{\sqrt[+]{t}} \frac{\sigma^n_{1,m} - \lambda}{\sqrt{\lambda - \rho^+_{1,m}}} \zeta^n_{1,m}(\lambda) \sqrt{\omega_{1,m}} \, \mathrm{d}t.$$

As $v_0 \in W \setminus (Z_{1,m} \cup E_{1,m})$, the function $v \mapsto \sqrt{\omega_{1,m}}$ is analytic for v near v_0. In all, we have shown that $\beta^n_{1,m}$ is analytic for v near v_0. Next let us prove that $\beta^n_{1,m}$ is continuous on W. By the previous considerations, $\beta^n_{1,m}$ is continuous at any potential of $W \setminus (Z_{1,m} \cup E_{1,m})$. By the estimate (8.71) it follows that $\beta^n_{1,m}$ is continuous at any potential in $E_{1,m}$. It thus remains to prove that $\beta^n_{1,m}$ is continuous at potentials of $Z_{1,m} \setminus E_{1,m}$. First we show that the restriction $\beta^n_{1,m}\big|_{Z_{1,m} \setminus E_{1,m}}$ is continuous. Indeed on $Z_{1,m}$, $\lambda^-_{1,m} = \tau_{1,m}$ and $(\sigma^n_{1,m} - \lambda)/w_{1,m}(\lambda) = 1$. Hence

$$\beta^n_{1,m}\big|_{Z_{1,m} \setminus E_{1,m}} = \int_{\tau_{1,m}}^{\mu_{1,m}} \zeta^n_{1,m}(\lambda) \, \mathrm{d}\lambda\big|_{Z_{1,m} \setminus E_{1,m}}.$$

Clearly, $\int_{\tau_{1,m}}^{\mu_{1,m}} \zeta^n_{1,m}(\lambda) \, \mathrm{d}\lambda\big|_{Z_{1,m} \setminus E_{1,m}}$ is continuous. Since $E_{1,m}$ is closed in W, it then remains to show that for any sequence $(v_k)_{k \geq 1} \subset W \setminus Z_{1,m} \cup E_{1,m}$ with $v_k \to v \in Z_{1,m} \setminus E_{1,m}$ as $k \to \infty$ one has $\lim_{k \to \infty} \beta^n_{1,m}(v_k) = \beta^n_{1,m}(v)$. Without loss of generality we may assume that $\inf_k |(\mu_{1,m} - \tau_{1,m})(v_k)| > 0$ and for any $k \geq 1$

$$|\lambda^+_{1,m}(v_k) - \mu_{1,m}(v_k)| \geq |\lambda^-_{1,m}(v_k) - \mu_{1,m}(v_k)|.$$

(Otherwise consider an appropriate subsequence of $(v_k)_{k\geq 1}$, and/or if necessary switch the roles of $\lambda^+_{1,m}$ and $\lambda^-_{1,m}$.)

In addition we may assume that $\sqrt[*]{\chi_p(\mu_{1,m}(v_k))}(=\delta(\mu_{1,m}(v_k)))$ coincides with $\sqrt[c]{\chi_p(\mu_{1,m}(v_k))}$ for any $k\geq 1$. Since $v\in Z_{1,m}\setminus E_{1,m}$ one has

$$\lim_{k\to\infty}\gamma_{1,m}(v_k)=0 \quad \text{and} \quad \lim_{k\to\infty}\omega_{1,m}(v_k)=\mu_{1,m}(v)-\tau_{1,m}(v)\neq 0.$$

Furthermore, since $v_k\notin E_{1,m}$ one has $\omega_{1,m}(v_k)=\mu_{1,m}(v_k)-\lambda^-_{1,m}(v_k)\neq 0$ for any $k\geq 1$. Hence for any $0<\epsilon<1$ there exists $k_\epsilon\geq 1$ so that

$$\Big|\frac{\gamma_{1,m}(v_k)}{\omega_{1,m}(v_k)}\Big|\leq\frac{\epsilon}{2}\qquad\forall\, k\geq k_\epsilon. \tag{8.82}$$

Making again the substitution $\lambda(t)=\lambda^-_{1,m}+t\omega_{1,m}$ in the integral defining $\beta^n_{1,m}\equiv\beta^n_{1,m}(v_k)$ one gets

$$\beta^n_{1,m}=\int_0^1\frac{\sigma^n_{1,m}-\lambda}{w_{1,m}(\lambda)}\zeta^n_{1,m}(\lambda)\omega_{1,m}\,\mathrm{d}t=I_\epsilon+II_\epsilon$$

where

$$I_\epsilon\equiv I_\epsilon(v_k)=\int_0^\epsilon\frac{\sigma^n_{1,m}-\lambda}{w_{1,m}(\lambda)}\zeta^n_{1,m}(\lambda)\omega_{1,m}\,\mathrm{d}t$$

and

$$II_\epsilon\equiv II_\epsilon(v_k)=\int_\epsilon^1\frac{\sigma^n_{1,m}-\lambda}{w_{1,m}(\lambda)}\zeta^n_{1,m}(\lambda)\omega_{1,m}\,\mathrm{d}t.$$

Taking into account the estimate (8.77) - (8.79) one sees that

$$|I_\epsilon(v_k)|\leq C\sqrt{\epsilon}\qquad\forall\, k\geq k_\epsilon$$

where $C>0$ is a constant independent of k. Next we estimate the integral $II_\epsilon(v_k)$. We claim that, according to the definition of the s-root,

$$w_{1,m}(\lambda)=(\tau_{1,m}-\lambda)\sqrt[+]{1-\frac{\gamma^2_{1,m}/4}{(\tau_{1,m}-\lambda)^2}} \tag{8.83}$$

for $\lambda=\lambda(t)$ with $\epsilon\leq t\leq 1$ and any v_k with $k\geq k_\epsilon$. Indeed note that for $\epsilon\leq t\leq 1$, and v_k with $k\geq k_\epsilon$, one has by (8.82), $|\omega_{1,m}/\gamma_{1,m}|\geq 2/\epsilon$ and hence

$$\big|t\frac{2\omega_{1,m}}{\gamma_{1,m}}\big|-1\geq\epsilon\cdot\frac{4}{\epsilon}-1=3\qquad\forall\,\epsilon\leq t\leq 1.$$

Therefore,

$$\frac{|\gamma_{1,m}|^2/4}{|\tau_{1,m}-\lambda(t)|^2}=\frac{|\gamma_{1,m}|^2/4}{|\gamma_{1,m}/2-t\omega_{1,m}|^2}\leq\frac{1}{(|t\frac{2\omega_{1,m}}{\gamma_{1,m}}|-1)^2}\leq\frac{1}{3^2}$$

yielding the claimed identity (8.83). Hence for any $k\geq k_\epsilon$

$$II_\epsilon(v_k)=\int_\epsilon^1\Big(1+\frac{\sigma^n_{1,m}-\tau_{1,m}}{\tau_{1,m}-\lambda}\Big)\Big(1-\frac{\gamma^2_{1,m}/4}{(\tau_{1,m}-\lambda)^2}\Big)^{-1/2}\zeta^n_{1,m}(\lambda)\omega_{1,m}\,\mathrm{d}t.$$

Since by Theorem 8.12, $\sigma^n_{1,m}-\tau_{1,m}=\gamma^2_{1,m}\ell^2_m$ locally uniformly on W, one has uniformly in $\epsilon\leq t\leq 1$

$$\frac{\sigma^n_{1,m}-\tau_{1,m}}{\tau_{1,m}-\lambda(t)}\Big|_{v_k}=\frac{\gamma^2_{1,m}\ell^2_m}{\tau_{1,m}-\lambda^-_{1,m}-t\omega_{1,m}}\Big|_{v_k}\xrightarrow[k\to\infty]{}0.$$

Here we used that by the definition of k_ϵ, one has for any $\epsilon\leq t\leq 1$ and any $k\geq k_\epsilon$

$$|\tau_{1,m}-\lambda^-_{1,m}-t\omega_{1,m}|_{v_k}\geq|\omega_{1,m}|(t-|\big|\frac{\gamma_{1,m}/2}{\omega_{1,m}}\big|)\big|_{v_k}\geq\frac{|\omega_{1,m}|}{2}\epsilon\big|_{v_k}$$

and $\lim_{k\to\infty}\omega_{1,m}(v_k) = \omega_{1,m}(v) = \mu_{1,m}(v) - \tau_{1,m}(v) \neq 0$. Similarly, one has uniformly for $\epsilon \leq t \leq 1$ and $k \geq k_\epsilon$,

$$\frac{\gamma_{1,m}}{\tau_{1,m} - \lambda(t)}\Big|_{v_k} = \frac{\gamma_{1,m}}{\tau_{1,m} - \lambda_{1,m}^- - t\omega_{1,m}}\Big|_{v_k} \xrightarrow[k\to\infty]{} 0.$$

As a consequence

$$\lim_{k\to\infty} II_\epsilon(v_k) = \int_0^1 \zeta_{1,m}^n(\lambda, v)\omega_{1,m}(v)\,\mathrm{d}t.$$

On the other hand, since $v \in Z_{1,m} \setminus E_{1,m}$,

$$\beta_{1,m}^n(v) = \int_0^1 \zeta_{1,m}^n(\lambda, v)\omega_{1,m}(v)\,\mathrm{d}t$$

and by the continuity of $\zeta_{1,m}^n(\lambda, v)$ in λ one concludes that

$$\beta_{1,m}^n(v) - \int_\epsilon^1 \zeta_{1,m}^n(\lambda, v)\omega_{1,m}(v)\,\mathrm{d}t = \int_0^\epsilon \zeta_{1,m}^n(\lambda, v)\omega_{1,m}(v)\,\mathrm{d}t = O(\epsilon).$$

Altogether we thus have shown that for any $\epsilon > 0$ there exists $k'_\epsilon \geq k_\epsilon$ so that

$$|\beta_{1,m}^n(v_k) - \beta_{1,m}^n(v)| \leq C\sqrt{\epsilon} \quad \forall\, k \geq k'_\epsilon,$$

where C can be chosen independently of ϵ and k. Since $\epsilon > 0$ can be chosen arbitrarily small it follows that $\lim_{k\to\infty}\beta_{1,m}^n(v_k) = \beta_{1,m}^n(v)$. This finishes the proof that $\beta_{1,m}^n$ is continuous at any point in v in $Z_{1,m} \setminus E_{1,m}$. It remains to check the weak analyticity of $\beta_{1,m}^n$ on $E_{1,m}$ and $Z_{1,m}$. On $E_{1,m}$ this is trivial since there the function $\beta_{1,m}^n$ vanishes identically. On $Z_{1,m}$ we can write

$$\beta_{1,m}^n = \int_{\tau_{1,m}}^{\mu_{1,m}} \varepsilon_{1,m}\zeta_{1,m}^n(\lambda)\,\mathrm{d}\lambda$$

where $\varepsilon_{1,m}$ is defined on $Z_{1,m}\setminus E_{1,m}$ by $\sqrt[*]{\chi_p(\mu_{1,m})} = \varepsilon_{1,m}\sqrt[c]{\chi_p(\mu_{1,m})}$, whereas on $Z_{1,m}\cap E_{1,m}$, $\varepsilon_{1,m} = 0$. Now consider a disc $D \subset Z_{1,m}$. As $E_{1,m}$ is an analytic subvariety, either $D \subset Z_{1,m} \cap E_{1,m}$ in which case $\beta_{1,m}^n\big|_D \equiv 0$ or $D \cap E_{1,m}$ is finite. Since $\int_{\tau_{1,m}}^{\mu_{1,m}} \zeta_{1,m}^n(\lambda)\,\mathrm{d}\lambda\big|_D$ is analytic (cf Lemma 6.21, Theorem 8.12, Lemma B.1) and $\beta_{1,m}^n$ is continuous on D, it follows that $\int_{\tau_{1,m}}^{\mu_{1,m}} \zeta_{1,m}^n(\lambda)\,\mathrm{d}\lambda\big|_D \equiv 0$ or $\epsilon_{1,m}\big|_{D\setminus E_{1,m}}$ is constant. In both cases it then follows that $\beta_{1,m}^n\big|_D$ is analytic. Thus we can apply Theorem A.6 to conclude that $\beta_{1,m}^n$ is analytic on W. Using that $\mu_{1,m}$, $\lambda_{1,m}^\pm$, and $\sigma_{1,m}^n$ are real valued on H_r^1 one sees that $\beta_{1,m}^n$ is real valued on H_r^1 as well. The proof for η_n is completely analoguous except for the fact that switching from $\lambda_{1,n}^-$ to $\rho_{1,n}^-$ may change its value by π (mod 2π) in view of (8.68). Hence we have $\eta_n = \int_{\rho_{1,n}^-}^{\mu_{1,n}} \frac{\psi_n(\lambda)}{\sqrt[*]{\chi_p(\lambda)}}\,\mathrm{d}\lambda$ (mod π). Then we proceed as in the proof for the analyticity of $\beta_{1,m}^n$, $m \neq n$, given above. To show that $\beta_{2,m}^n$ is real analytic on W for any $m \in \mathbb{Z}$ one proceeds as in the case of $\beta_{1,m}^n$, studying $\beta_{2,m}^n$ on $W \setminus (Z_{2,m} \cup E_{2,m})$, $Z_{2,m}$, and $E_{2,m}$. Recall that

$$\beta_{2,m}^n = \int_{\lambda_{2,m}^-}^{\mu_{1,m}} \frac{\psi_n(-\frac{1}{16\lambda})}{\sqrt[*]{\chi_p(-\frac{1}{16\lambda})}}\frac{\mathrm{d}\mu}{16\,\mu^2}.$$

By (8.80)-(8.81),

$$\frac{\psi_n(-\frac{1}{16\mu})}{\sqrt[c]{\chi_p(-\frac{1}{16\mu})}} = \frac{1}{\pi_n}\frac{\sigma_{2,m}^n - \mu}{w_{2,m}(-\frac{1}{16\mu})}\zeta_{2,m}^n(-\frac{1}{16\mu}) = \frac{1}{\pi_n}\frac{\sigma_{2,m}^n - \mu}{\sqrt{(\lambda_{2,m}^+ - \mu)(\lambda_{2,m}^- - \mu)}}\zeta_{2,m}^n(-\frac{1}{16\mu})$$

where

$$\zeta_{2,m}^n(-\frac{1}{16\mu}) = \mathrm{i}\frac{\psi_{n,1}(-\frac{1}{16\mu})}{\sqrt[c]{\chi_1(-\frac{1}{16\mu})}/\sqrt[c]{\chi_1(0)}}\frac{1}{\psi_{n,2}(\infty)}\Big(\prod_{k\neq m}\frac{\sigma_{2,k}^n - \mu}{w_{2,k}(-\frac{1}{16\mu})}\Big).$$

Since for any given $v_0 \in W$, $\zeta_{2,m}^n(-\frac{1}{16\mu})$ as a function of μ, is analytic on $\tilde{U}_{2,m} \times V_{v_0}$ and satisfies the estimate $\zeta_{2,m}^n(-\frac{1}{16\mu}) = O(1)$ one can analyze the analyticity in the same way as in the case of $\beta_{1,m}^n$, the main difference being that the integrand of $\beta_{2,m}^n$ has the additional factor $\frac{1}{16\mu^2}$. □

The results established so far allow to prove the following main result of this section.

***Theorem* 8.31** *For any $n \geq 0$, the series $\sum_{m\neq n} \beta^n_{1,m}$ and $\sum_{m\in\mathbb{Z}} \beta^n_{2,m}$ converge locally uniformly on W to real analytic funcions on W, $\beta^n_1 := \sum_{m\neq n} \beta^n_{1,m}$, $\beta^n_2 := \sum_{m\in\mathbb{Z}} \beta^n_{2,m}$, such that $\beta^n_1 = o(1)$ and $\beta^n_2 = O(\frac{1}{n})$ as $n \to \infty$. The angle function*

$$\theta_n = \eta_n + \beta^n_1 + \beta^n_2 = \eta_n + \sum_{m\neq n} \beta^n_{1,m} + \sum_{m\in\mathbb{Z}} \beta^n_{2,m},$$

defined modulo 2π, is a real valued function on $H^1_r \setminus Z_n$ and extends to a real analytic function on $W \setminus Z_n$ when taken modulo π.

Proof. By Lemma 8.28 and the Cauchy-Schwarz inequality,

$$\begin{aligned}\sum_{m\neq n} |\beta^n_{1,m}| &= \sum_{0<|m-n|\leq n/2} |\beta^n_{1,m}| + \sum_{|m-n|>n/2} |\beta^n_{1,m}| \\ &\leq C\Big(\sum_{|m|\geq n/2} |\gamma_{1,m}|^2 + |\mu_{1,m} - \tau_{1,m}|^2\Big)^{1/2} + C\big(\|(\gamma_{1,m})_{m\in\mathbb{Z}}\| + \|(\mu_{1,m} - \tau_{1,m})_{m\in\mathbb{Z}}\|\big)\Big(\sum_{k>n/2} \frac{1}{k^2}\Big)^{1/2}.\end{aligned}$$

Since $(\gamma_{1,m})_{m\in\mathbb{Z}} \in \ell^2$ and $(\mu_{1,m} - \tau_{1,m})_{m\in\mathbb{Z}} \in \ell^2$, the two latter sums converge to zero as n tends to infinity. It also follows from Lemma 8.28 that

$$\sum_{m\in\mathbb{Z}} |\beta^n_{2,m}| \leq C\frac{1}{n}\big(\|(\gamma_{2,m})_{m\in\mathbb{Z}}\| + \|(\mu_{2,m} - \tau_{2,m})_{m\in\mathbb{Z}}\|\big).$$

Furthermore, by Lemma 8.29, $\beta^n_{j,m}$ is real analytic on W for any $(j,m) \neq (1,n)$. Using Theorem A.4 one sees that $\sum_{m\neq n} \beta^n_{1,m}$ and $\sum_{m\in\mathbb{Z}} \beta^n_{2,m}$ are real analytic on W as well. The claim about θ_n, $n \geq 0$, then follows from the properties of η_n stated in Lemma 8.29. □

Remark 8.32. In view of the identities (8.67) and (8.68), the vlaues of θ_n can be explicitly computed for potentials $v \in W$ satisfying

$$\mu_k(v) \in \{\lambda^-_k(v),\, \lambda^+_k(v)\} \quad \forall\, k \in \mathbb{Z}. \tag{8.84}$$

If in addition for any given $n \in \mathbb{Z}$, $v \in W \setminus Z_n$ with $\mu_n(v) = \lambda^-_n(v)$, then $\theta_n(v) = 0 \pmod{2\pi}$ whereas if $v \in W \setminus Z_n$ with $\mu_n(v) = \lambda^+_n(v)$, then $\theta_n(v) = \pi \pmod{2\pi}$.

Finally we introduce for any $n \geq 1$ the angle variable η_{-n} on $W \setminus Z_{-n}$. For any $v \in W \setminus Z_{-n}$ we define

$$\theta_{-n}(v) := \eta_{-n}(v) + \sum_{m\in\mathbb{Z}} \beta^{-n}_{1,m}(v) + \sum_{m\neq -n} \beta^{-n}_{2,m}(v) \tag{8.85}$$

where

$$\eta_{-n}(v) \equiv \beta^{-n}_{2,-n}(v) := \int_{\lambda^-_{2,-n}}^{\mu_{2,-n}} \frac{\psi_{-n}(-\frac{1}{16\mu})}{\sqrt[*]{\chi_p(-\frac{1}{16\mu})}} \frac{\mathrm{d}\mu}{16\mu^2} \pmod{2\pi} \tag{8.86}$$

$$\beta^{-n}_{2,m}(v) := \int_{\lambda^-_{2,m}}^{\mu_{2,m}} \frac{\psi_{-n}(-\frac{1}{16\mu})}{\sqrt[*]{\chi_p(-\frac{1}{16\mu})}} \frac{\mathrm{d}\mu}{16\mu^2} \quad \forall\, m \neq n \tag{8.87}$$

$$\beta^{-n}_{1,m}(v) := \int_{\lambda^-_{1,m}}^{\mu_{1,m}} \frac{\psi_{-n}(\lambda)}{\sqrt[*]{\chi_p(\lambda)}} \lambda \quad \forall\, m \in \mathbb{Z} \tag{8.88}$$

and $\psi_{-n}(\lambda, v)$ is given by Corollary 8.15,

$$\psi_{-n}(\lambda, q, p) = \psi_n(\frac{1}{16\lambda}, -q, p)\frac{1}{16\lambda^2}.$$

One can analyze the above integrals in the same way as the ones appearing in the definition of $\theta_n(v)$ with $n \geq 0$. But in fact, one does not need to since these integrals can be expressed in terms of corresponding integrals of $\theta_n(-q, p)$.

***Lemma* 8.33** *For any $n \geq 1$, one has on $W \setminus Z_{-n}$*

$$\begin{aligned} \eta_{-n}(q,p) &= -\,\eta_n(-q,p) + \pi \pmod{2\pi} && (8.89)\\ \beta^{-n}_{2,m}(q,p) &= -\,\beta^{n}_{1,-m}(-q,p) \quad \forall\ m \neq -n && (8.90)\\ \beta^{-n}_{1,m}(q,p) &= -\,\beta^{n}_{2,-m}(-q,p) \quad \forall\ m \in \mathbb{Z}. && (8.91) \end{aligned}$$

Hence $\theta_{-n}(q,p) = -\theta_n(-q,p) + \pi \pmod{2\pi}$.

Proof. To prove (8.89) recall that $\lambda^-_{-n}(q,p) \neq \lambda^+_{-n}(q,p)$ iff $\lambda^-_n(-q,p) \neq \lambda^+_n(-q,p)$, $\psi_{-n}(\lambda,q,p) = \frac{1}{16\lambda^2}\psi_n(\frac{1}{16\lambda},-q,p)$, $\delta(\lambda,q,p) = \delta(\frac{1}{16\lambda}.-q,p)$ and $\sqrt[*]{\chi_p(\lambda,q,p)} = \sqrt[*]{\chi_p(\frac{1}{16\lambda},-q,p)}$. Substituting the expression for ψ_{-n} into the definition of η_{-n} and using that $-\mu_{2,-n} = \mu_{2,n}$ and $-\lambda^-_{2,-n} = \lambda^+_{2,n}$ one gets

$$\eta_{-n} = \int_{(-16\lambda^-_{2,-n})^{-1}}^{(-16\mu_{2,-n})^{-1}} \frac{\psi_{-n}(\lambda)}{\sqrt[*]{\chi_p(\lambda)}}\,\mathrm{d}\lambda = \int_{(16\lambda^+_{2,n})^{-1}}^{(16\mu_{2,n})^{-1}} \frac{\psi_n(\frac{1}{16\lambda},-q,p)}{\sqrt[*]{\chi_p(\lambda,q,p)}}\frac{\mathrm{d}\lambda}{16\lambda^2}.$$

Since $\lambda^+_{2,n}(q,p) = \lambda^+_{1,n}(-q,p)$, $\mu_{2,n}(q,p) = \mu_{1,n}(-q,p)$ and $\sqrt[*]{\chi_p(\lambda,q,p)} = \sqrt[*]{\chi_p(\frac{1}{16\lambda},-q,p)}$ one gets

$$\eta_{-n} = \int_{(16\lambda^+_{1,n}(-q,p))^{-1}}^{(16\mu_{1,n}(-q,p))^{-1}} \frac{\psi_n(\frac{1}{16\lambda},-q,p)}{\sqrt[*]{\chi_p(\frac{1}{16\lambda},-q,p)}}\frac{\mathrm{d}\lambda}{16\lambda^2}$$

with the change of variables $\mu = \frac{1}{16\lambda}$ one then arrives at

$$\eta_{-n} = -\int_{\lambda^+_{1,n}(-q,p)}^{\mu_{2,n}(-q,p)} \frac{\psi_n(\mu,-q,p)}{\sqrt[*]{\chi_p(\mu,-q,p)}}\,\mathrm{d}\mu.$$

Finally we use that $\int_{\lambda^-_{1,n}(-q,p)}^{\lambda^+_{1,n}(-q,p)} \frac{\psi_n(\mu)}{\sqrt[*]{\chi_p(\mu)}}\,\mathrm{d}\mu = \pi \pmod{2\pi}$ to conclude that

$$\eta_{-n}(v) = -\eta_n(-q,p) + \pi.$$

The identities (8.90) and (8.91) are proved in the same way. □

Remark 8.34. Sometimes it will be useful to express the bounds of the integrals defining $\beta^n_{j,m}$ in terms of $\lambda^\pm_m$, $-\lambda^\pm_m$, μ_m, $-\mu_m$:

$$\begin{aligned} \beta^n_{1,m} &= \int_{\lambda^-_m}^{\mu_m} \frac{\psi_n(\lambda)}{\sqrt[*]{\chi_p(\lambda)}}\,\mathrm{d}\lambda \quad \forall\ m \geq 0,\ \forall\ n \geq 0\\ \beta^n_{1,-m} &= \int_{-\lambda^+_m}^{-\mu_m} \frac{\psi_n(\lambda)}{\sqrt[*]{\chi_p(\lambda)}}\,\mathrm{d}\lambda \quad \forall\ m \geq 1,\ \forall\ n \geq 0\\ \beta^n_{2,m} &= \int_{-\lambda^+_{-m}}^{-\mu_{-m}} \frac{\psi_n(\lambda)}{\sqrt[*]{\chi_p(\lambda)}}\,\mathrm{d}\lambda \quad \forall\ m \geq 0,\ \forall\ n \geq 0\\ \beta^n_{2,-m} &= \int_{\lambda^-_{-m}}^{\mu_{-m}} \frac{\psi_n(\lambda)}{\sqrt[*]{\chi_p(\lambda)}}\,\mathrm{d}\lambda \quad \forall\ m \geq 1,\ \forall\ n \geq 0 \end{aligned}$$

and

$$\begin{aligned} \beta^{-n}_{1,m} &= \int_{\lambda^-_m}^{\mu_m} \frac{\psi_{-n}(\lambda)}{\sqrt[*]{\chi_p(\lambda)}}\,\mathrm{d}\lambda \quad \forall\ m \geq 0,\ \forall\ n \geq 1\\ \beta^{-n}_{1,-m} &= \int_{-\lambda^+_m}^{-\mu_m} \frac{\psi_{-n}(\lambda)}{\sqrt[*]{\chi_p(\lambda)}}\,\mathrm{d}\lambda \quad \forall\ m \geq 1,\ \forall\ n \geq 1\\ \beta^{-n}_{2,m} &= \int_{-\lambda^+_{-m}}^{-\mu_{-m}} \frac{\psi_{-n}(\lambda)}{\sqrt[*]{\chi_p(\lambda)}}\,\mathrm{d}\lambda \quad \forall\ m \geq 0,\ \forall\ n \geq 1\\ \beta^{-n}_{2,-m} &= \int_{\lambda^-_{-m}}^{\mu_{-m}} \frac{\psi_{-n}(\lambda)}{\sqrt[*]{\chi_p(\lambda)}}\,\mathrm{d}\lambda \quad \forall\ m \geq 1,\ \forall\ n \geq 1. \end{aligned}$$

9 Birkhoff coordinates

In this chapter we construct Birkhoff coordinates for the sinh-Gordon equation. A key ingredient are the actions and angles introduced and studied in Chapter 8.

9.1 Definition of Birkhoff coordinates and their regularity

Recall that in section 8.1 we have introduced on $\hat{W}$ the action variables

$$I_n(v) = -\frac{1}{\pi}\int_{\Gamma_n} \frac{1}{\lambda} F(\lambda, v)\, \mathrm{d}\lambda, \quad n \in \mathbb{Z}. \tag{9.1}$$

They are real analytic and by Proposition 8.6(ii), satisfy on $\hat{W}$

$$I_{-n}(q,p) = I_n(-q,p) \quad \forall\, n \in \mathbb{Z}. \tag{9.2}$$

Furthermore, I_n is nonnegative on H^1_r. On the other hand, in Section 8.3 we have introduced for any $n \in \mathbb{Z}$ the angle variable θ_n on $W \setminus Z_n$ where $W \subset \hat{W}$ is a neighborhood of H^1_r which like $\hat{W}$ is invariant under the map $\mathcal{S}_{rec} : (q,p) \mapsto (-q,p)$ and Z_n is the subvariety $Z_n = \{\, v \in \hat{W} \,:\, \lambda_n^+(v) = \lambda_n^-(v) \,\}$. The variable θ_n is defined mod 2π and is real analytic on $W \setminus Z_n$ when considered mod π. We recall from Lemma 8.33 that for $n \geq 1$ and $(q,p) \in W \setminus Z_{-n}$, $(-q,p)$ is in $W \setminus Z_n$ and

$$\theta_{-n}(q,p) := -\theta_n(-q,p) + \pi \pmod{2\pi}. \tag{9.3}$$

Furthermore, the additive factor π in (9.3) implies that Remark 8.32 continues to hold for negative n. Indeed if $v = (q,p) \in W \setminus Z_{-n}$ satisfies $\mu_k(v) \in \{\lambda_k^-(v), \lambda_k^+(v)\}$ and $\mu_{-n}(v) = \lambda_{-n}^-(v)$, then in view of the symmetries established for the μ_k's and the $\lambda_k^\pm$'s one has $\mu_k(-q,p) \in \{\lambda_k^-(-q,p), \lambda_k^+(-q,p)\}$ $\forall\, k \in \mathbb{Z}$ and $\mu_n(-q,p) = \lambda_n^+(-q,p)$ implying that $\theta_n(-q,p) = \pi \pmod{2\pi}$ (cf Remark 8.32) and in turn $\theta_{-n}(q,p) = 0 \pmod{2\pi}$. Similarly, if $\mu_{-n}(v) = \lambda_{-n}^+(v)$ instead of $\mu_{-n}(v) = \lambda_{-n}^-(v)$, then $\theta_n(-q,p) = 0$ $\pmod{2\pi}$ and therefore $\theta_{-n}(q,p) = \pi \pmod{2\pi}$.

In order to obtain coordinates on H^1_r we consider the associated Birkhoff coordinates. Recall that on H^1_r, $I_n \geq 0$ for any $n \in \mathbb{Z}$, hence

$$x_n = \sqrt[+]{2I_n}\cos(\theta_n), \quad y_n = \sqrt[+]{2I_n}\sinh(\theta_n)$$

are well defined on $H^1_r \setminus Z_n$. To extend x_n, y_n, $n \in \mathbb{Z}$, to a common neighborhood of H^1_r requires some careful analysis. First we note that for any $(q,p) \in H^1_r \setminus Z_{-n}$ with $n \geq 1$, one has $I_{-n}(q,p) = I_n(-q,p)$ and, $\theta_{-n}(q,p) = -\theta_n(-q,p) + \pi$, implying that

$$x_{-n}(q,p) = -x_n(-q,p), \quad y_{-n}(q,p) = y_n(-q,p).$$

Hence in a first step we focus our attention on the case $n \geq 0$. Recall from Theorem 8.8 that

$$I_n = \frac{1}{\tau_n}(\xi_n\gamma_n)^2.$$

Note that on H^1_r, $\tau_n > 0 \;\; \forall\, n \geq 0$ and since $\tau_n = n\pi + \ell^2_n$ as $n \to \infty$ it follows, maybe after shrinking $\hat{W}$, if needed, that $\mathrm{Re}\tau_n > 0$ on $\hat{W}$ for any $n \geq 0$. Hence $\sqrt[+]{\tau_n}$ on $\hat{W}$ for any $n \geq 0$. Hence $\sqrt[+]{\tau_n}$ is well defined on $\hat{W}$ for any $n \geq 0$ and so is

$$\sqrt[+]{2I_n} = \frac{2}{\sqrt[+]{2I_n}}\xi_n\gamma_n.$$

Furthermore, one has asymptotics

$$\sqrt[+]{2\tau_n} = \sqrt[+]{2n\pi} + \frac{1}{\sqrt{n}}\ell^2_n \quad \text{as } n \to \infty.$$

For any $n \geq 0$, we define on $W \setminus Z_n$

$$x_n := \frac{\xi_n}{\sqrt[+]{2\tau_n}}\left(e^{\mathrm{i}\beta_n} z_n^+ + e^{-\mathrm{i}\beta_n} z_n^-\right)$$

$$y_n := \frac{\xi_n}{\mathrm{i}\,\sqrt[+]{2\tau_n}}\left(e^{\mathrm{i}\beta_n} z_n^+ - e^{-\mathrm{i}\beta_n} z_n^-\right)$$

where

$$z_n^{\pm} := \gamma_n e^{\pm \mathrm{i}\eta_n}, \quad \beta_n := \beta_1^n + \beta_2^n. \tag{9.4}$$

By Theorem 8.31, β_n and hence $e^{\pm \mathrm{i}\beta_n}$ are analytic on W. By Theorem 8.8, ξ_n is real analytic on $\hat{W}$. On the other hand γ_n is not even continuous on W and η_n is real analytic on $W \setminus Z_n$ when considered mod π. In a first step we prove that z_n^+, z_n^- are analytic on $W \setminus Z_n$.

***Lemma* 9.1** *For any $n \geq 0$, the functions $z_n^{\pm} = \gamma_n e^{\pm \mathrm{i}\eta_n}$ are analytic on $W \setminus Z_n$.*

Proof. Sine on $W \setminus Z_n$, λ_n^+ and λ_n^- are simple periodic eigenvalues there exist locally around every point in $W \setminus Z_n$ analytic functions ρ_n^+, ρ_n^- such that the set equality $\{\rho_n^-, \rho_n^+\} = \{\lambda_n^-, \lambda_n^+\}$ holds. Let

$$\tilde{\gamma}_n := \rho_n^+ - \rho_n^-, \quad \tilde{\eta}_n := \int_{\rho_n^-}^{\mu_n} \frac{\psi_n(\lambda)}{\sqrt[*]{\chi_p(\lambda)}} \, \mathrm{d}\lambda.$$

Depending on whether $\rho_n^+ = \lambda_n^+$ or $\rho_n^+ = \lambda_n^-$ we have $\gamma_n = \tilde{\gamma}_n$, $\eta_n = \tilde{\eta}_n$ or in view of (8.68)

$$\gamma_n = -\tilde{\gamma}_n, \quad \eta_n = \int_{\lambda_n^-}^{\mu_n} \frac{\psi_n(\lambda)}{\sqrt[*]{\chi_p(\lambda)}} \, \mathrm{d}\lambda = \tilde{\eta}_n + \pi \pmod{2\pi}.$$

In either case,

$$\gamma_n e^{\pm \mathrm{i}\eta_n} = \tilde{\gamma}_n e^{\pm \mathrm{i}\tilde{\eta}_n}.$$

The right hand side of the latter identity is analytic. □

Next we study the limiting behaviour of $z_n^{\pm}$ as v approaches a potential v_0 in Z_n. This limit is different from zero when v_0 is in the set

$$Y_n := \{ \, v \in W : \; \mu_n \notin G_n \, \}.$$

Note that Y_n is an open subset of W and that $Y_n \cap H_r^1 = \emptyset$ since for $v \in H_r^1$, $\mu_n \in G_n$. On Y_n, the sign function

$$\varepsilon_n := \frac{\sqrt[*]{\chi_p(\mu_n)}}{\sqrt[c]{\chi_p(\mu_n)}} \tag{9.5}$$

is well defined and locally constant where we recall that $\sqrt[*]{\chi_p(\mu_n)} = \delta(\mu_n)$.

***Lemma* 9.2** *Let $n \geq 0$. If $v \in Y_n \setminus Z_n$ tends to $v_0 \in W \cap Z_n$ then*

$$\gamma_n e^{\pm \mathrm{i}\eta_n} \to \begin{cases} 2(\tau_n - \mu_n)(1 \pm \varepsilon_n) e^{\chi_n} & \text{if } v_0 \in W \cap Z_n \cap Y_n \\ 0 & \text{if } v_0 \in W \cap Z_n \setminus Y_n \end{cases}$$

where $\chi_n \equiv \chi_n(v_0)$ is defined as

$$\chi_n := \int_{\tau_n}^{\mu_n} \frac{\zeta_n(\tau_n) - \zeta_n(\lambda)}{\lambda - \tau_n} \, \mathrm{d}\lambda$$

and

$$\zeta_n(\lambda) := \Big(\prod_{m \neq n} \frac{\sigma_{1,m}^n - \lambda}{w_{1,m}(\lambda)} \Big) \frac{\psi_{n,2}(\lambda)/\psi_{n,2}(\infty)}{\sqrt[c]{\chi_2(\lambda)}/\sqrt[c]{\chi_2(\infty)}}.$$

Proof. Let $n \geq 0$. By the product representations of $\sqrt[c]{\chi_p(\lambda)}$ (cf (6.52), (6.56), (6.58)) and $\psi_n(\lambda)$ (Theorem 8.12) one has for $v \in Y_n \setminus Z_n$

$$\frac{\psi_n(\lambda)}{\sqrt[*]{\chi_p(\lambda)}} = \mathrm{i}\varepsilon_n \frac{\zeta_n(\lambda)}{w_{1,n}(\lambda)}$$

and hence η_n can be written as

$$\mathrm{i}\eta_n = \mathrm{i} \int_{\lambda_n^-}^{\mu_n} \frac{\psi_n(\lambda)}{\sqrt[*]{\chi_p(\lambda)}} \, \mathrm{d}\lambda = -\varepsilon_n \int_{\lambda_n^-}^{\mu_n} \frac{\zeta_n(\lambda)}{w_{1,n}(\lambda)} \, \mathrm{d}\lambda$$

where the root $w_{1,n}(\lambda)$ is well defined along the path of integration chosen in such a way that it meets G_n only in its initial point λ_n^-. We decompose the numerator $\zeta_n(\lambda)$ into three terms

$$\zeta_n(\lambda) = (\zeta_n(\lambda) - \zeta_n(\tau_n)) + (\zeta_n(\tau_n) - 1) + 1$$

and denote the corresponding integrals by o_n, υ_n, and ω_n. respectively. The limit of o_n is straightforward by Lemma 6.18 and Theorem 8.12, ζ_n is analytic on $U_n \times V_{v_0}$ where $V_{v_0} \subset W$ is a sufficiently small neighborhood of v_0 and U_n the n'th isolating neighborhood working for any potential in V_{v_0}. If $v \to v_0$ then $w_{1,n}(\lambda, v) \to \tau_n(v_0) - \lambda$, $\lambda_n^\pm(v) \to \tau_n(v_0)$, and $\mu_n(v) \to \mu_n(v_0)$. Thus by the definition of $\chi_n(v)$,

$$-o_n = -\int_{\lambda_n^-}^{\mu_n} \frac{\zeta_n(\lambda) - \zeta_n(\tau_n)}{w_{1,n}(\lambda)}\,\mathrm{d}\lambda \to -\int_{\tau_n}^{\mu_n} \frac{\zeta_n(\lambda) - \zeta_n(\tau_n)}{\tau_n - \lambda}\,\mathrm{d}\lambda = \chi_n(v_0).$$

For the second term we have

$$\upsilon_n = (\zeta_n(\tau_n) - 1)\int_{\lambda_n^-}^{\mu_n} \frac{\mathrm{d}\lambda}{w_{1,n}(\lambda)}. \tag{9.6}$$

By Lemma 9.3, $\zeta_n(\tau_n) - 1 = O(\gamma_n)$. Considering the estimate of $\int_{\lambda_n^-}^{\mu_n} \frac{\mathrm{d}\lambda}{w_{1,n}(\lambda)}$ we may assume without loss of generality that along the sequence of potentials v in $Y_n \setminus Z_n$ converging to v_0 one has $|\mu_n - \lambda_n^-| \le |\mu_n - \lambda_n^+|$. Otherwise switch the roles of λ_n^- and λ_n^+. The substitution $\lambda(t) = \lambda_n^- + t(\mu_n - \lambda_n^-)$ then leads to

$$\int_{\lambda_n^-}^{\mu_n} \frac{\mathrm{d}\lambda}{w_{1,n}(\lambda)} = (\mu_n - \lambda_n^-)\int_0^1 \frac{1}{w_{1,n}(\lambda(t))}\,\mathrm{d}t = O\big(|\mu_n - \lambda_n^-|\int_0^1 \frac{1}{|t(\mu_n - \lambda_n^-)|^{1/2}}\frac{1}{|\lambda_n^+ - \lambda(t)|^{1/2}}\,\mathrm{d}t\big).$$

Since $|\mu_n - \lambda_n^-| \le |\mu_n - \lambda_n^+|$ one has $|\lambda_n^+ - \lambda(t)| \ge |\gamma_n|/2$ for any $0 \le t \le 1$ implying that

$$\int_{\lambda_n^-}^{\mu_n} \frac{1}{w_{1,n}(\lambda)}\,\mathrm{d}\lambda = O(|\mu_n - \lambda_n^-|^{1/2}|\gamma_n|^{-1/2}).$$

Altogether it then follows that

$$\upsilon_n = (\zeta_n(\tau_n) - 1)\int_{\lambda_n^-}^{\mu_n} \frac{\mathrm{d}\lambda}{w_{1,n}(\lambda)} = O(|\gamma_n|^{1/2}|\mu_n - \lambda_n^-|^{1/2}) \to 0$$

as v tends to v_0. Now consider the term $\omega_n = \int_{\lambda_n^-}^{\mu_n} \frac{\mathrm{d}\lambda}{w_{1,n}(\lambda)}$. We compute it on $Y_n \setminus Z_n$ mod $2\pi\mathrm{i}$ by choosing the straight line path $\lambda(t) = \tau_n + t\gamma_n/2$ with t in $[-1, \rho_n]$ and $\rho_n := 2(\mu_n - \tau_n)/\gamma_n \in \mathbb{C} \setminus [-1, 1]$. In the case the interval $[\lambda_n^-, \mu_n]$ intersects $G_n \setminus \{\lambda_n^-\}$, it actually contains all of G_n. One easily verifies that in this case, the choice of the sign of $w_{1,n}(\lambda)$ along G_n does not matter. We then get mod $2\pi\mathrm{i}$

$$\omega_n = \int_{\lambda_n^-}^{\mu_n} \frac{\mathrm{d}\lambda}{\sqrt[s]{(\lambda_n^+ - \lambda)(\lambda_n^- - \lambda)}} = \int_{-1}^{\rho_n} \frac{\mathrm{d}t}{\sqrt[s]{(1-t)(-1-t)}}$$

where by the definition of the standard root

$$\sqrt[s]{(1-t)(-1-t)} = -t\sqrt[+]{1-t^2}, \quad |t| \to \infty.$$

We claim that

$$e^{-\varepsilon_n\omega_n} = -\rho_n + \varepsilon_n\sqrt[s]{(1-\rho_n)(-1-\rho_n)}. \tag{9.7}$$

Indeed, both sides as functions of ρ_n, are solutions of the initial value problem

$$\frac{f'(z)}{f(z)} = \frac{-\varepsilon_n}{\sqrt[s]{(1-z)(-1-z)}}, \quad f(-1) = 1.$$

Now consider the limit $v \to v_0$. First let us treat the case where $v_0 \in Y_n \cap Z_n$. Then $\mu_n - \tau_n$ does not converge to zero. This implies that $\rho_n^{-1} = \frac{\gamma_n}{2(\mu_n - \tau_n)} \to 0$ and

$$\begin{aligned}\gamma_n e^{-\varepsilon_n\omega_n} =& -\gamma_n\rho_n + \varepsilon_n\gamma_n\sqrt[s]{(1-\rho_n)(-1-\rho_n)} = -\gamma_n\rho_n + \varepsilon_n\gamma_n(-\rho_n)\sqrt[+]{1-\rho_n^{-2}} \\ =& 2(\tau_n - \mu_n)(1 + \varepsilon_n\sqrt[+]{1-\rho_n^{-2}}) \to 2(\tau_n - \mu_n)(1 + \varepsilon_n).\end{aligned}$$

In the case where $v_0 \in W \cap Z_n \setminus Y_n$, one has $\gamma_n\rho_n \to 0$ as $v \to v_0$ and thus concludes that

$$\gamma_n e^{-\varepsilon_n\omega_n} = 2(\tau_n - \mu_n) + \varepsilon_n\gamma_n\sqrt[s]{(1-\rho_n)(-1-\rho_n)} \to 0.$$

Note that if $v_0 \in W \cap Z_n \setminus Y_n$, $2(\tau_n - \mu_n)(1+\varepsilon_n) = 0$ independent of the value of $\varepsilon_n \in \{1, -1\}$, and hence this case can be included in the one where $\rho_n^{-1} \to 0$. Combining the results for all three integrals we obtain

$$\gamma_n e^{\mathrm{i}\eta_n} = \gamma_n e^{-\varepsilon_n \omega_n} e^{-\varepsilon_n(o_n + v_n)} \to 2(\tau_n - \mu_n)(1+\varepsilon_n) e^{\varepsilon_n \chi_n}.$$

This limit is zero for $\tau_n = \mu_n$ and for $\tau_n \neq \mu_n$ with $\varepsilon_n = -1$. Hence we can drop ε_n in $e^{\varepsilon_n \chi_n}$ and obtain the formula $2(\tau_n - \mu_n)(1+\varepsilon_n)e^{\chi_n}$. In the same way one shows that

$$\gamma_n e^{-\mathrm{i}\eta_n} \to 2(\tau_n - \mu_n)(1-\varepsilon_n) e^{-\varepsilon_n \chi_n}$$

which vanishes for $\tau_n - \mu_n = 0$ or for $\tau_n - \mu_n \neq 0$ and $\varepsilon_n = 1$. Hence we can replace $e^{-\varepsilon_n \chi_n}$ by e^{χ_n}. □

Lemma 9.3 *Let $n \geq 0$. Then for $\lambda \in G_n$,*

$$\zeta_n(\lambda) = 1 + O(\gamma_n)$$

locally uniformly on W. In more detail, the claimed estimate means that for any $v \in W$ there exists $C > 0$ and a neighborhood V_v of v in W so that on V_v, $\sup_{\lambda \in G_n} |\zeta_n(\lambda) - 1| \leq C|\gamma_n|$.

Proof. Let $n \geq 0$. Similarly as in the proof of Lemma 9.2 we write

$$\frac{\psi_n(\lambda)}{\sqrt[c]{\chi_p(\lambda)}} = -\frac{\zeta_n(\lambda)}{\mathrm{i} w_n(\lambda)}, \quad \zeta_n(\lambda) = \Big(\prod_{m \neq n} \frac{\sigma_{1,m}^n - \lambda}{w_{1,m}(\lambda)}\Big) \frac{\psi_{n,2}(\lambda)/\psi_{n,2}(\infty)}{\sqrt[c]{\chi_2(\lambda)}/\sqrt[c]{\chi_2(\infty)}}.$$

Integrating over $\Gamma_n (\equiv \Gamma_{1,n})$ we obtain by Theorem 8.12 for any $\tau \in G_n$

$$1 = -\frac{1}{2\pi \mathrm{i}} \int_{\Gamma_n} \frac{\zeta_n(\lambda)}{w_{1,n}(\lambda)} \,\mathrm{d}\lambda = -\frac{1}{2\pi \mathrm{i}} \int_{\Gamma_n} \frac{\zeta_n(\tau)}{w_{1,n}(\lambda)} \,\mathrm{d}\lambda + \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_n} \frac{\zeta_n(\tau) - \zeta_n(\lambda)}{w_{1,n}(\lambda)} \,\mathrm{d}\lambda.$$

Since $-\frac{1}{2\pi \mathrm{i}} \int_{\Gamma_n} \frac{1}{w_{1,n}(\lambda)} \,\mathrm{d}\lambda = 1$ we get

$$1 = \zeta_n(\tau) + \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_n} \frac{\zeta_n(\tau) - \zeta_n(\lambda)}{w_{1,n}(\lambda)} \,\mathrm{d}\lambda = \zeta_n(\tau) + O(\max_{\lambda \in G_n} |\zeta_n(\lambda) - \zeta_n(\tau)|)$$

where for the last line we used Lemma 8.18. By Corollary 6.24, $\prod_{m \neq n} \frac{\sigma_{1,m}^n - \lambda}{w_{1,m}(\lambda)}$ is bounded on U_n locally uniformly in v and uniformly in n. Since also $\psi_{n,2}(\lambda)$ and $\sqrt[c]{\chi_2(\lambda)}$ are analytic for λ in U_n and locally uniformly bounded in v, it follows that the same is true for $\dot{\zeta}_n(\lambda) = \partial_\lambda \zeta_n(\lambda)$ with $\lambda \in G_n$ by Cauchy's estimate. Therefore

$$\max_{\lambda \in G_n} |\zeta_n(\lambda) - \zeta_n(\tau)| \leq \max_{\lambda \in G_n} |\dot{\zeta}_n(\lambda)| |\gamma_n| = O(\gamma_n)$$

locally uniformly in v and uniformly in n. This proves the claimed statement. □

We now extend the functions $z_n^\pm$ on all of W as follows

$$z_n^\pm := \begin{cases} 2(\tau_n - \mu_n)(1 \pm \varepsilon_n) e^{\chi_n} & \text{if } v \in W \cap Z_n \cap Y_n \\ 0 & \text{if } v \in W \cap Z_n \setminus Y_n \end{cases} \tag{9.8}$$

with χ_n as in Lemma 9.2. To establish that $z_n^\pm$ are analytic on W we need the following asymptotic estimates.

Lemma 9.4 *For any $n \geq 0$*

$$z_n^\pm = O(|\gamma_n| + |\mu_n - \tau_n|)$$

locally uniformly on W and uniformly in $n \geq 0$.

Proof. Let $n \geq 0$. Form the proof of Lemma 9.2 one sees that on $Y_n \setminus Z_n$,

$$z_n^+ = \gamma_n e^{\mathrm{i}\eta_n} = (-\gamma_n \rho_n + \varepsilon_n \gamma_n \sqrt[s]{(1-\rho_n)(-1-\rho_n)}) e^{-\varepsilon_n(v_n + o_n)}$$

where ε_n is given by (9.5) and $\rho_n = 2(\mu_n - \tau_n)/\gamma_n$. In case where $2|\mu_n - \tau_n| \leq |\gamma_n|$, i.e. $|\rho_n| \leq 1$,

$$|-\gamma_n \rho_n + \varepsilon_n \gamma_n \sqrt[s]{(1-\rho_n)(-1-\rho_n)}| \leq 3|\gamma_n|, \tag{9.9}$$

while in the case $2|\mu_n - \tau_n| > |\gamma_n|$, i.e. $|\rho_n| > 1$ one has

$$z_n^+ = \gamma_n e^{\mathrm{i}\eta_n} = 2(\tau_n - \mu_n)(1 + \varepsilon_n \sqrt[+]{1 - \rho_n^{-2}})e^{-\varepsilon_n(\upsilon_n + o_n)}$$

where

$$|2(\tau_n - \mu_n)(1 + \varepsilon_n) \sqrt[+]{1 - \rho_n^{-2}})| \le 6|\mu_n - \tau_n|. \tag{9.10}$$

The exponential term $e^{-\varepsilon_n(\upsilon_n + o_n)}$ is locally uniformly bounded (cf Corollary 6.24). So we get on $Y_n \setminus Z_n$

$$z_n^+ = O(|\gamma_n| + |\mu_n - \tau_n|). \tag{9.11}$$

By Lemma 9.2, estimate (9.10) continues to hold on $Y_n \cap Z_n$. Furthermore one easily verifies that (9.9) is also valid on $W \setminus Y_n$ for any choice of $\varepsilon_n \in \{1, -1\}$. Hence (9.11) holds in a locally uniform fashion on all of W. In the same way one proves the claimed estimate for $z_n^- = \gamma_n e^{-\mathrm{i}\eta_n}$. □

***Proposition* 9.5** *For any $n \ge 0$, the functions z_n^+, z_n^- are analytic on W.*

Proof. Let $n \ge 0$. We apply Theorem A.6 to the functions z_n^+, z_n^- on the domain W with the subvariety $W \cap Z_n$. By Lemma 9.1, these functions are analytic on $W \setminus Z_n$. Next we show that they are continuous at each potential in Z_n. First note that it follows from the formulas of z_n^+, z_n^- on Z_n, that the restrictions of z_n^+, z_n^- to Z_n are continuous. Approaching a potential in Z_n from within $Y_n \setminus Z_n$, the corresponding values of z_n^+, z_n^- converge by Lemma 9.2 to the ones of the limiting potential. On the other hand, approaching a potential in Z_n from $W \setminus (Y_n \cup Z_n)$, one has $|\mu_n - \tau_n| \le |\gamma_n|$ and hence $z_n^\pm = \gamma_n e^{\pm\mathrm{i}\eta_n}$ converges to zero by Lemma 9.4. Altogether we have shown that the functions z_n^+, z_n^- are continuous on W. In order to appy Theorem A.6 it remains to show that the restrictions of z_n^+, z_n^- to Z_n are weakly analytic. Let D be a one-dimensional complex disc contained in Z_n. If the center of D is in Y_n, then the entire disc D is in Y_n if chosen sufficiently small. The analyticity of $z_n^\pm = \gamma_n e^{\pm\mathrm{i}\eta_n}$ on D is then evident from the formula (9.8), the definition of χ_n (cf Lemma 9.2) and the local constancy of ε_n on Y_n. If the center of D does not belong to Y_n, then consider the analytic function $\mu_n - \tau_n$ on D. This function either vanishes identically on D in which case $z_n^\pm$ vanishes identically also by (9.8). Or it vanishes in only finitely many points. Outside these points, D is in Y_n, hence z_n^+, z_n^- are analytic there. By continuity and analytic continuation, these functions are analytic on all of D. □

We are now ready to define the Birkhoff coodinates on W for any $n \ge 0$

$$x_n = \frac{\xi_n}{\sqrt[+]{2\tau_n}}\big(z_n^+ e^{\mathrm{i}\beta_n} + z_n^- e^{-\mathrm{i}\beta_n}\big), \quad y_n = \frac{\xi_n}{\mathrm{i}\sqrt[+]{2\tau_n}}\big(z_n^+ e^{\mathrm{i}\beta_n} - z_n^- e^{-\mathrm{i}\beta_n}\big), \tag{9.12}$$

and for $n \ge 1$ and $v = (q, p)$

$$x_{-n}(q, p) := -x_n(-q, p), \quad y_{-n}(q, p) := y_n(-q, p).$$

To define the corresponding Birkhoff map we introduce the weighted ℓ^2-sequence spaces $\ell^{2,\alpha}$ for any $\alpha \in \mathbb{R}$,

$$\ell^{2,\alpha} := \{\ x = (x_k)_k \in \ell^2(\mathbb{Z}, \mathbb{C}) :\ \|x\|_\alpha = \big(\sum_{k \in \mathbb{Z}} \langle k \rangle^{2\alpha} |x_k|^2\big)^{1/2} < \infty\ \}$$

and set $\ell^{2,\alpha}_{\mathbb{R}} := \ell^{2,\alpha} \cap \ell^2_{\mathbb{R}}$, as well as $h^\alpha_c := \ell^{2,\alpha} \times \ell^{2,\alpha}$, $\quad h^\alpha_r := \ell^{2,\alpha}_{\mathbb{R}} \times \ell^{2,\alpha}_{\mathbb{R}}$.

***Theorem* 9.6** *The map $\Phi(v) := \big((x_n(v))_{n\in\mathbb{Z}}, (y_n(v))_{n\in\mathbb{Z}}\big)$, defined for v in W, takes values in $h_c^{1/2}$ and $\Phi : W \to h_c^{1/2}$ is real analytic. Furthermore, for any $s \ge 0$, $\Phi(W \cap H_r^{1+s}) \subset h_r^{s+1/2}$.*

Proof. By Proposition 9.5 z_n^+, z_n^- are analytic on W for any $n \ge 0$ and admit the estimate $z_n^\pm = O(|\gamma_n| + |\mu_n - \tau_n|)$ locally uniformly on W and uniformly in n. By Theorem 8.8, ξ_n is real analytic on W and satisfies the estimate $\xi_n = 1 + \ell_n^2$ as $n \to \infty$ locally uniformly on W. Furthermore by Lemma 3.16, $\mu_n = n\pi + \ell_n^2$ as $n \to \infty$ and by Lemma 3.17, $\tau_n = n\pi + \ell_n^2$, $\gamma_n = \ell_n^2$ as $n \to \infty$ locally uniformly on W. Since $\mathrm{Re}(\tau_n) > 0$ for any $n \ge 0$ locally uniformly on W it follows that $(\tau_n)^{-1/2} = \frac{1}{\sqrt{n\pi}} + \frac{1}{\langle n \rangle^{3/2}}\ell_n^2$ as $n \to \infty$. Finally, by Theorem 8.31, β_n and hence $e^{\mathrm{i}\beta_n}$ are analytic on W and bounded locally uniformly on W and uniformly in n. Altogether it follows that x_n and y_n are analytic on W and satisfy the estimate

$$|x_n|, |y_n| = O\big(\frac{1}{\langle n \rangle^{1/2}}|\gamma_n| + \frac{1}{\langle n \rangle^{1/2}}|\mu_n - \tau_n|\big) \tag{9.13}$$

locally uniformly on W and uniformly in n and $(x_n)_{n\geq 0}$, $(y_n)_{n\geq 0}$ are in $\ell^{2,1/2}(\mathbb{Z}_{\geq 0}, \mathbb{C})$. By the definition of x_{-n}, y_{-n} for $n \geq 1$ it then follows that $\big((x_n)_n, (y_n)_n\big) \in h_c^{1/2}$ locally uniformly on W. It then follows from Theorem A.5 that $\Phi : W \to h_c^{1/2}$ is an analytic map. Going trough the arguments of the proof one verifies that $\Phi(H_r^1) \subset h_r^{1/2}$ and hence Φ is real analytic. By Theorem 4.10 and the estimate 9.13 it follows that $\Phi(H_r^{1+s}) \subset h_r^{s+1/2}$ for any $s \geq 0$. □

9.2 Canonical relations

In this section we compute Poisson brackets between various coordinate functions. The results will be used to analyze the differential of the Birkhoff map on H_r^1.

First le us conisider the action variables $I_n (= 4J_{0,n})$, $n \in \mathbb{Z}$, defined on $\hat{W}$ (cf Section 8.1). We prove that the Poisson brackets $\{I_n, I_m\}$ are well defined and vanish for any $n, m \in \mathbb{Z}$. Actually, it turns out that the action variables $J_{k,n}$ on all levels Poisson commute with each other.

***Proposition* 9.7** *For any $n, m, k, l \in \mathbb{Z}$, the Poisson brackets $\{J_{k,n}, J_{l,m}\}$ is well defined on $\hat{W}$ and vanishes identically,*

$$\{J_{k,n}, J_{l,m}\} = 0.$$

Proof. By Proposition 8.6, for any $n, k \in \mathbb{Z}$, $J_{k,n}$ is analytic on $\hat{W}$ and its L^2-gradient is given by

$$\partial J_{k,n} = -\frac{1}{\pi} \int_{\Gamma_n} \frac{1}{\lambda} (4\lambda)^k \frac{\partial \Delta(\lambda)}{\sqrt[c]{\chi_p(\lambda)}} \, \mathrm{d}\lambda.$$

By Lemma 7.2, $\partial\Delta(\lambda)$ and hence $\partial J_{k,n}$ are in L_c^2. As a consequence, the Poisson brackets $\{J_{k,n}, J_{l,m}\}$ are well defined and one has

$$\{J_{k,n}, J_{l,m}\} = \frac{16}{\pi^2} \int_{\Gamma_n} \int_{\Gamma_m} (4\lambda)^{k-1} (4\mu)^{l-1} \frac{\{\Delta_\lambda, \Delta_\mu\}}{\sqrt[c]{\chi_p(\mu)} \sqrt[c]{\chi_p(\lambda)}} \, \mathrm{d}\lambda \, \mathrm{d}\mu.$$

By Theorem 7.4, $\{J_{k,n}, J_{l,m}\} = 0$ on $\hat{W}$. □

Proposition 9.7 can be extended in the following way:

***Proposition* 9.8** *For any differentiable function G on H_r^1 which at every point v depends only on the periodic spectrum of $Q(v)$, the Poisson bracket $\{G, \Delta_\lambda\}$ is well defined on H_r^1 for any $\lambda \in \mathbb{C}^*$ and vanishes identically. As a consequence, for any $k \in \mathbb{Z}$, $\{G, I_k\}$ is well defined on H_r^1 and vanishes as well.*

Proof. Since for any $\lambda \in \mathbb{C}^*$, $\partial\Delta_\lambda$ is in L_c^2 (Lemma 8.2), $\{G, \Delta_\lambda\}$ is well defined on H_r^1. Furthermore $X := JP^{-1}\partial\Delta_\lambda$ is an analytic (Hamiltonian) vector field on H_c^1 (cf Lemma 8.2, Theorem 2.2). Since the Lie derivative $L_X \Delta_\mu$ is given by $\langle \partial\Delta_\mu, X\rangle_r = \{\Delta_\mu, \Delta_\lambda\}$ it follows from Theorem 7.4 that $L_X \Delta_\mu = 0$. Hence the periodic spectrum is constant along the flow lines of X. But then also G is constant along them since G is assumed to depend only on the periodic spectrum. It follows that $L_X G = 0$ and hence $\{G, \Delta_\lambda\} = 0$. Using that for any $k \in \mathbb{Z}$, $\partial I_k = -\frac{4}{\pi} \int_{\Gamma_n} \frac{1}{\lambda} \frac{\partial\Delta(\lambda)}{\sqrt[c]{\chi_p(\lambda)}} \, \mathrm{d}\lambda$ one concludes that $\{G, I_k\}$ is well defined on H_r^1 and vanishes identically. □

Next we want to compute the Poisson brackets between action and angle variables. Recall that for any $n \in \mathbb{Z}$, $Z_n = \{ v \in \hat{W} \ : \ \gamma_n(v) = 0 \}$. Furthermore, in Section 8.2 we have introduced the open neighborhood $W \subset \hat{W}$ of H_r^1 with the property that $\mathcal{S}_{rec}(W) = W$ (cf Section 6.4) and for any $n \in \mathbb{Z}$, the angle θ_n is defined on $W \setminus Z_n$,

$$\theta_n = \eta_n + \sum_{m \neq n} \beta_{1,m}^n + \sum_{m \in \mathbb{Z}} \beta_{2,m}^n. \tag{9.14}$$

A key ingredient for computing the brackets between actions and angles are the formulas for the brackets $\{\mu_n, \Delta_\lambda\}$ of Lemma 7.8. Since we want to apply later Lemma D.1 (interpolation lemma) it is useful to rewrite these formulas in terms of the notations introduced in Sections 6.3, 6.4, and 8.2. Recall that by Lemma 7.8 for any $m \in \mathbb{Z}$ and $\lambda \neq \pm\mu_m$, the following identities hold on $\hat{W}$.

$$8\{\mu_m, \Delta_\lambda\} = \frac{\delta(\mu_m)}{\dot{\chi}_D(\mu_m)} \lambda \chi_D(\lambda) \frac{2\mu_m}{\lambda^2 - \mu_m^2} = \frac{\delta(\mu_m)}{\dot{\chi}_D(\mu_m)} \lambda \chi_D(\lambda) \Big(\frac{1}{\lambda - \mu_m} - \frac{1}{\lambda + \mu_m}\Big).$$

Furthermore, we have listed the Dirichlet eigenvalues for $v \in \hat{W}$ in terms of the bi-infinite sequences $(\mu_{1,n})_n$, $(\mu_{2,n})_2$ (cf Section 6.3) and introduced $\kappa_{2,n} = (-16\mu_{2,n})^{-1}$. The Dirichlet eigenvalues are then $(\mu_{1,n})_n$ and $(\kappa_{2,n})_n$. It is straightforward to verify that (9.3) can be written as follows: for any $m \in \mathbb{Z}$ and $v \in \hat{W}$,

$$8\{\mu_{1,m}, \Delta_\lambda\} = \frac{\delta(\mu_{1,m})}{\dot{\chi}_D(\mu_{1,m})}\lambda \cdot (\frac{\chi_D(\lambda)}{\lambda - \mu_{1,m}} - \frac{\chi_D(\lambda)}{\lambda + \mu_{1,m}}) \tag{9.15}$$

and

$$8\{\kappa_{2,m}, \Delta_\lambda\} = \frac{\delta(\kappa_{2,m})}{\dot{\chi}_D(\kappa_{2,m})}\lambda \cdot (\frac{\chi_D(\lambda)}{\lambda - \kappa_{2,m}} - \frac{\chi_D(\lambda)}{\lambda + \kappa_{2,m}}). \tag{9.16}$$

First we need to establish the following

***Lemma* 9.9** *For any $n \in \geq 0$ and $\lambda \in \mathbb{C}^*$, $\{\theta_n, \Delta_\lambda\}$ is well defined on $W \setminus Z_n$ and on this set*

$$8\,\{\theta_n, \Delta_\lambda\} = \lambda \cdot (\psi_n(\lambda) - \psi_n(-\lambda)).$$

Proof. Let $n \geq 0$. By Lemma 8.29, $\beta^n_{1,m}$ $(m \neq n)$, $\beta^n_{2,m}$ $(m \in \mathbb{Z})$ are analytic on W whereas η_n is analytic on $W \setminus Z_n$ when considered modulo π. Furthermore, $\sum_{m\neq n} \beta^n_{1,m}$, $\sum_{m\in\mathbb{Z}} \beta^n_{2,m}$ converge locally uniformly on W and hence $\theta_n = \eta_n + \sum_{m\neq n} \beta^n_{1,m} + \sum_{m\in\mathbb{Z}} \beta^n_{2,m}$ is analytic on $W \setminus Z_n$ when considered modulo π. Since by Lemma 8.2, $\partial\Delta_\lambda$ is in L^2_c for any $\lambda \in \mathbb{C}^*$, the brackets $\{\theta_n, \Delta_\lambda\}$, $\{\eta_n, \Delta_\lambda\}$, $\{\beta^n_{1,m}, \Delta_\lambda\}$, and $\{\beta^n_{2,m}, \Delta_\lambda\}$ are well defined (on their domains of definition) and

$$\{\theta_n, \Delta_\lambda\} = \{\eta_n, \Delta_\lambda\} + \sum_{m\neq n} \{\beta^n_{1,m}, \Delta_\lambda\} + \sum_{m\in\mathbb{Z}} \{\beta^n_{2,m}, \Delta_\lambda\}.$$

Let us compute $\{\beta^n_{1,m}, \Delta_\lambda\}$ on H^1_r for any $m \neq n$. First consider $v \in H^1_r$ with $\lambda^-_{1,m} < \mu_{1,m} < \lambda^+_{1,m}$. Recall that by (8.64), $\beta^n_{1,m}$ is given by

$$\beta^n_{1,m} = \int_{\lambda^-_{1,m}}^{\mu_{1,m}} \frac{\psi_n(\lambda)}{\sqrt[*]{\chi_p(\lambda)}}\, \mathrm{d}\lambda. \tag{9.17}$$

The lower bound of the latter integral and its integrand are spectral invariants, hence invariant under the Hamiltonian vector field $X_\lambda = JP^{-1}\partial\Delta_\lambda$ defined on H^1_c (cf Proposition 9.8). Since the bracket with Δ_λ amounts to a differentiation in direction X_λ one gets by Leibniz rule

$$\{\beta^n_{1,m}, \Delta_\lambda\} = \frac{\psi_n(\mu_{1,m})}{\sqrt[*]{\chi_p(\mu_{1,m})}}\{\mu_{1,m}, \Delta_\lambda\}$$

which, when combined with formula (9.15), yields

$$8\{\beta^n_{1,m}, \Delta_\lambda\} = \frac{\psi_n(\mu_{1,m})}{\sqrt[*]{\chi_p(\mu_{1,m})}} \frac{\delta(\mu_{1,m})}{\dot{\chi}_D(\mu_{1,m})}\lambda \cdot (\frac{\chi_D(\lambda)}{\lambda - \mu_{1,m}} - \frac{\chi_D(\lambda)}{\lambda + \mu_{1,m}}).$$

By the definition of the $*$-root, one has $\sqrt[*]{\chi_p(\mu_{1,m})} = \delta(\mu_{1,m})$ and hence

$$\frac{8}{\lambda}\{\beta^n_{1,m}, \Delta_\lambda\} = \frac{\psi_n(\mu_{1,m})}{\dot{\chi}_D(\mu_{1,m})}(\frac{\chi_D(\lambda)}{\lambda - \mu_{1,m}} - \frac{\chi_D(\lambda)}{\lambda + \mu_{1,m}}). \tag{9.18}$$

This identity holds on all of H^1_r since by Corollary 7.15, $\lambda^-_{1,m} < \mu_{1,m} < \lambda^+_{1,m}$ holds on an open dense subset of H^1_r, and $\chi_D(\lambda)/(\lambda \pm \mu_{1,m})$ continuously extends to $\lambda = \pm\mu_{1,m}$ implying that both sides of the identity (9.18) are continuous on H^1_r. By analyticity, identity (9.18) actually holds on W. Since when computing derivatives we can ignore the modulus π part in the definition of η_n, it then follows again by the same arguments that on $W \setminus Z_n$

$$\frac{8}{\lambda}\{\eta_n, \Delta_\lambda\} = \frac{\psi_n(\mu_{1,n})}{\dot{\chi}_D(\mu_{1,n})}(\frac{\chi_D(\lambda)}{\lambda - \mu_{1,n}} - \frac{\chi_D(\lambda)}{\lambda + \mu_{1,n}}).$$

To compute $\{\beta^n_{2,m}, \Delta_\lambda\}$, $m \in \mathbb{Z}$, one argues as in the case of $\beta^n_{1,m}$. After a change of variable $\lambda = -\frac{1}{16\mu}$ in the definition of $\beta^n_{2,m}$ one gets

$$\beta^n_{2,m} = \int_{(-16\lambda^-_{2,m})^{-1}}^{\kappa_{2,m}} \frac{\psi_n(\lambda)}{\sqrt[*]{\chi_p(\lambda)}}\, \mathrm{d}\lambda.$$

Arguing as above one then concludes that on W

$$\frac{8}{\lambda}\{\beta^n_{2,m}, \Delta_\lambda\} = \frac{\psi_n(\kappa_{2,m})}{\dot\chi_D(\kappa_{2,m})}\Big(\frac{\chi_D(\lambda)}{\lambda-\kappa_{2,m}} - \frac{\chi_D(\lambda)}{\lambda+\kappa_{2,m}}\Big).$$

Altogether one concludes that on $W \setminus Z_n$, $\frac{8}{\lambda}\{\theta_n, \Delta_\lambda\} = I_{n,\lambda} - II_{1,\lambda}$ where

$$I_{n,\lambda} := \sum_{m\in\mathbb{Z}} \frac{\psi_n(\mu_{1,m})}{\dot\chi_D(\mu_{1,m})}\frac{\chi_D(\lambda)}{\lambda-\mu_{1,m}} + \frac{\psi_n(\kappa_{2,m})}{\dot\chi_D(\kappa_{2,m})}\frac{\chi_D(\lambda)}{\lambda-\kappa_{2,m}}$$

$$II_{n,\lambda} := \sum_{m\in\mathbb{Z}} \frac{\psi_n(\mu_{1,m})}{\dot\chi_D(\mu_{1,m})}\frac{\chi_D(\lambda)}{\lambda+\mu_{1,m}} + \frac{\psi_n(\kappa_{2,m})}{\dot\chi_D(\kappa_{2,m})}\frac{\chi_D(\lambda)}{\lambda+\kappa_{2,m}}.$$

To evaluate $I_{n,\lambda}$ we want to apply Lemma D.1 (interpolation) with $\varphi(\lambda) = \psi_n(\lambda)$ and $\sigma_{1,k} = \mu_{1,k}$, $\sigma_{2,k} = \mu_{2,k}$, $k \in \mathbb{Z}$. Clearly $\mu_{1,k}$, $\mu_{2,k}$ satisfy (D.1) - (D.2). Furthermore, $\psi_n(\lambda) = -\frac{1}{\pi_n}\psi_{n,1}(\lambda)\psi_{n,2}(\lambda)/\psi_{n,2}(\infty)$ where $\psi_{n,1}(\lambda) = \prod_{k\neq n}\frac{\sigma^n_{1,k}-\lambda}{\pi_k}$, and $\psi_{n,2}(\lambda)/\psi_{n,2}(\infty)$ is analytic near $\lambda = \infty$, it follows from Lemma B.5 that

$$\sup_{\lambda\in\partial B_N} \Big|\frac{\psi_n(\lambda)}{\sin(\lambda)}\Big| = O\left(\frac{1}{N}\right) \quad \text{as } N \to \infty.$$

Similarly, since $\psi_{n,2}(\lambda) = \prod_{k\in\mathbb{Z}}\frac{\sigma^n_{2,k}+\frac{1}{16\lambda}}{\pi_k}$ we have again by Lemma B.5 that

$$\sup_{\lambda\in\partial B_{-N}} \Big|\frac{\psi_n(\lambda)}{\sin(-\frac{1}{16\lambda})}\Big| = \sup_{\mu\in\partial B_N}\Big|\frac{\psi_n(-\frac{1}{16\mu})}{\sin(\mu)}\Big| = O(1).$$

Hence Lemma D.1 applies and we have $I_{n,\lambda} = \psi_n(\lambda)$. Similarly we argue fot $II_{n,\lambda}$: changing the index of summation in $II_{n,\lambda}$ to $-m$ and using that $\mu_{1,-m} = -\mu_{1,m}$ ($\forall\ m \neq 0$), $\kappa_{2,-m} = -\kappa_{2,m}$ ($\forall\ m \neq 0$), and $\mu_{1,0} = -\kappa_{2,0}$ one gets

$$II_{n,\lambda} = \sum_{m\in\mathbb{Z}} \frac{\psi_n(-\mu_{1,m})}{\dot\chi_D(-\mu_{1,m})}\frac{\chi_D(\lambda)}{\lambda-\mu_{1,m}} + \frac{\psi_n(-\kappa_{2,m})}{\dot\chi_D(-\kappa_{2,m})}\frac{\chi_D(\lambda)}{\lambda-\kappa_{2,m}}.$$

Since $\dot\chi_D$ is even one then arrives at

$$II_{n,\lambda} = \sum_{m\in\mathbb{Z}} \frac{\psi_n(-\mu_{1,m})}{\dot\chi_D(\mu_{1,m})}\frac{\chi_D(\lambda)}{\lambda-\mu_{1,m}} + \frac{\psi_n(-\kappa_{2,m})}{\dot\chi_D(\kappa_{2,m})}\frac{\chi_D(\lambda)}{\lambda-\kappa_{2,m}}.$$

We now apply Lemma D.1 to $\psi_n(-\lambda)$ to conclude that $II_{n,\lambda} = \psi_n(-\lambda)$. Altogether we have shown that on W,

$$8\{\theta_n, \Delta_\lambda\} = \lambda\cdot(\psi_n(\lambda) - \psi_n(-\lambda))$$

for any $\lambda \in \mathbb{C}^*$. □

***Proposition* 9.10** *For any $n, k \in \mathbb{Z}$, $\{\theta_n, I_k\}$ is well defined on $W \setminus Z_n$ and*

$$\{\theta_n, I_k\} = -\delta_{n,k}.$$

Proof. By Lemma 7.2, $\partial\Delta_\lambda$ and hence $\partial I_k = -\frac{4}{\pi}\int_{\Gamma_k}\frac{1}{\lambda}\frac{\partial\Delta(\lambda)}{\sqrt[c]{\chi_p(\lambda)}}\,\mathrm{d}\lambda$ is in L^2_c for any $v \in W$. Hence $\{\theta_n, I_k\}$ is well defined on $W \setminus Z_n$ and

$$\{\theta_n, I_k\} = -\frac{4}{\pi}\int_{\Gamma_k}\frac{1}{\lambda}\frac{\{\theta_n, \Delta(\lambda)\}}{\sqrt[c]{\chi_p(\lambda)}}\,\mathrm{d}\lambda.$$

By Lemma 9.9 it then follows that

$$\{\theta_n, I_k\} = -\frac{1}{2\pi}\int_{\Gamma_k}\frac{8}{\lambda}\{\theta_n, \Delta(\lambda)\}\frac{\mathrm{d}\lambda}{\sqrt[c]{\chi_p(\lambda)}} = -\frac{1}{2\pi}\int_{\Gamma_k}\frac{\psi_n(\lambda)}{\sqrt[c]{\chi_p(\lambda)}}\,\mathrm{d}\lambda + \frac{1}{2\pi}\int_{\Gamma_k}\frac{\psi_n(-\lambda)}{\sqrt[c]{\chi_p(\lambda)}}\,\mathrm{d}\lambda.$$

By Theorem 8.12, $\frac{1}{2\pi}\int_{\Gamma_k}\frac{\psi_n(\lambda)}{\sqrt[c]{\chi_p(\lambda)}}\,\mathrm{d}\lambda = \delta_{n,k}$. To evaluate $\frac{1}{2\pi}\int_{\Gamma_k}\frac{\psi_n(-\lambda)}{\sqrt[c]{\chi_p(\lambda)}}\,\mathrm{d}\lambda$ make the change of variable $\lambda \mapsto \mu := -\lambda$ and use that $\sqrt[c]{\chi_p(-\lambda)} = -\sqrt[c]{\chi_p(\lambda)}$ (Lemma 6.21) to conclude that again by Theorem 8.12

$$\frac{1}{2\pi}\int_{\Gamma_k}\frac{\psi_n(-\lambda)}{\sqrt[c]{\chi_p(\lambda)}}\,\mathrm{d}\lambda = \frac{1}{2\pi}\int_{\Gamma_{k^-}}\frac{\psi_n(\mu)}{\sqrt[c]{\chi_p(\mu)}}\,\mathrm{d}\mu = 0.$$

Altogether we have shown that for any $n \geq 0$, $k \in \mathbb{Z}$, $\{\theta_n, I_k\} = -\delta_{n,k}$ on $W \setminus Z_n$. It remains to consider the Posson brackets $\{\theta_{-n}, I_k\}$ on $W \setminus Z_{-n}$ for $n \geq 1$, $k \in \mathbb{Z}$. By Lemma 6.6, $\gamma_{-n}(q,p) \neq 0$ iff $\gamma_n(-q,p) \neq 0$ implying that $(q,p) \in W \setminus Z_{-n}$ iff $(-q,p) \in W \setminus Z_n$. Furthermore recall from Section 9.1 that $\theta_{-n}(q,p)$ is defined by $\theta_{-n}(q,p) = -\theta_n(-q,p) + \pi \pmod{2\pi}$. Hence $(\partial_q\theta_{-n}, \partial_p\theta_{-n})\big|_{(q,p)} = -(-\partial_q\theta_n, \partial_p\theta_n)\big|_{(-q,p)}$. On the other hand, by Proposition 8.6, $I_k(q,p) = I_{-k}(-q,p)$ and hence

$$(\partial_q I_k, \partial_p I_k)\big|_{(q,p)} = (-\partial_q I_{-k}, \partial_p I_{-k})\big|_{(-q,p)}.$$

Therefore

$$\{\theta_{-n}, I_k\}(q,p) = \int_0^1 \begin{pmatrix} \partial_q\theta_n \\ -\partial_p\theta_n \end{pmatrix} \cdot JP^{-1} \begin{pmatrix} -\partial_q I_{-k} \\ \partial_p I_{-k} \end{pmatrix} \mathrm{d}x\big|_{(-q,p)} = \{\theta_n, I_{-k}\}(-q,p) = -\delta_{-n,k}$$

as claimed. □

Propositions 9.7 and 9.10 yield the following applications.

***Corollary* 9.11** *For any $m, k \in \mathbb{Z}$, $\{x_m, I_k\}$ and $\{y_m, I_k\}$ are well defined on W and*

$$\{x_m, I_k\} = \delta_{m,k} y_m, \quad \{y_m, I_k\} = -\delta_{mk} x_m.$$

Proof. Let $m, k \in \mathbb{Z}$. Note that x_m, y_m are analytic on W and ∂I_k is in L^2_c on W. Hence $\{x_m, I_k\}$ and $\{y_m, I_k\}$ are well defined on W. For v in $H^1_r \setminus Z_m$, x_m and y_m are given by $x_m = \sqrt[+]{2I_m}\cos(\theta_m)$ and $y_m = \sqrt[+]{2I_m}\sin(\theta_m)$ implying that

$$\{x_m, I_k\} = \frac{1}{\sqrt[+]{2I_m}}\cos(\theta_m)\{I_m, I_k\} - \sqrt[+]{2I_m}\sin(\theta_m)\{\theta_m, I_k\}.$$

By Proposition 9.7 and 9.10 one then concludes that

$$\{x_m, I_k\} = \sqrt[+]{2I_m}\sin(\theta_m)\delta_{mk} = y_m\delta_{mk}.$$

Similarly one shows that $\{y_m, I_k\} = -x_m\delta_{mk}$ on $H^1_r \setminus Z_m$. Since the quantities involved are continuous on H^1_r and $H^1_r \setminus Z_m$ is dense in H^1_r it follows that these identities hold on H^1_r. By analyticity they then hold on W. □

We finish this section with computing the Poisson brackets of the coordinate functions x_n, y_n, x_m, y_m for potentials in $H^1_r \cap Z_n \cap Z_m$ with $n, m \in \mathbb{Z}$ arbitrary. First we need to study the gradients of $z_n^\pm$ on $Z_n \cap H^1_r$. We do this by approximating a potential $v_0 \in Z_n \cap H^1_r$ by potentials v in

$$V_n^{mid} := \{\ v \in H^1_r \ :\ \mu_n = \tau_n,\ \operatorname{sign}(\delta(\mu_n)) = (-1)^{n-1}\ \}.$$

Such an approximation is possible in view of Proposition 6.10 and Proposition 7.14. We recall that $\partial\mu_n \in L^2_r$ on H^1_r (Lemma 7.7), $\partial\tau_n \in L^2_r$ on H^1_r (formula (6.11) and Lemma 7.2), and $\partial\lambda_n^\pm$ on $H^1_r \setminus Z_n$ (Lemma 7.2) and that by Theorem 2.2, these functions are continuous when considered as maps with values in L^2_r.

***Lemma* 9.12** *For any $n \in \mathbb{Z}$ and $v_0 \in Z_n \cap H^1_r$, $\partial z_n^\pm \in L^2_c$ and*

$$\partial z_n^\pm = 2(\partial\tau_n - \partial\mu_n) \pm \mathrm{i} \lim_{\substack{v\to v_0 \\ v\in V_n^{mid}}} (\partial\lambda_n^+ - \partial\lambda_n^-)$$

where the limit is to be understood in the L^2_r sense and the two terms converge individually.

Proof. Let us first consider the case $n \geq 0$. As in Lemma 9.2 define

$$\zeta_n(\lambda) = \Big(\prod_{m\neq n} \frac{\sigma^n_{1,m} - \lambda}{w_{1,m}(\lambda)}\Big) \frac{\psi_{n,2}(\lambda)/\psi_{n,2}(\infty)}{\sqrt[c]{\chi_2(\lambda)}/\sqrt[c]{\chi_2(\infty)}}$$

and recall that

$$\psi_n(\lambda) = -\frac{1}{\pi_n}\Big(\prod_{m\neq n} \frac{\sigma^n_{1,m} - \lambda}{\pi_m}\Big)\psi_{n,2}(\lambda)/\psi_{n,2}(\infty).$$

Since $\operatorname{sign}(\psi_n(\mu_n)) = (-1)^{n-1}$ and $\operatorname{sign}(\delta(\mu_n)) = (-1)^{n-1}$ one has (taking into account that $\sqrt[*]{\chi_p(\mu_n)} = \delta(\mu_n)$)

$$\frac{\psi_n(\mu_n)}{\sqrt[*]{\chi_p(\mu_n)}} = \frac{\zeta_n(\mu_n)}{\sqrt[+]{(\lambda_n^+ - \mu_n)(\mu_n - \lambda_n^-)}}$$

and hence

$$\eta_n = \int_{\lambda_n^-}^{\mu_n} \frac{\zeta_n(\lambda)}{\sqrt[+]{(\lambda_n^+ - \lambda)(\lambda - \lambda_n^-)}}\, \mathrm{d}\lambda \pmod{2\pi}.$$

Furthermore, all arguments in the proof of Lemma 9.2 remain valid for $\lambda \in G_n$ if we set $\varepsilon_n = 1$ and replace $-\mathrm{i}w_{1,n}(\lambda)$ by the positive root $\sqrt[+]{(\lambda_n^+ - \lambda)(\lambda - \lambda_n^-)}$. We write $\eta_n = \eta_n' + \eta_n'' \pmod{2\pi}$ where

$$\eta_n' = \int_{\lambda_n^-}^{\mu_n} \frac{\zeta_n(\lambda_n^-)}{\sqrt[+]{(\lambda_n^+ - \lambda)(\lambda - \lambda_n^-)}}\, \mathrm{d}\lambda, \quad \eta_n'' = \int_{\lambda_n^-}^{\mu_n} \frac{\zeta_n(\lambda) - \zeta_n(\lambda_n^-)}{\sqrt[+]{(\lambda_n^+ - \lambda)(\lambda - \lambda_n^-)}}\, \mathrm{d}\lambda$$

and decompose $z_n^+ = \gamma_n e^{\mathrm{i}\eta_n}$ into the product

$$z_n^+ = (\gamma_n e^{\mathrm{i}\eta_n'}) e^{\mathrm{i}\eta_n''}$$

where by the arguments used in the proof of Lemma 9.2, both factors in the latter product are analytic on W. Note that $\gamma_n e^{\mathrm{i}\eta_n'} = 0$ on $Z_n \cap H_r^1$ (cf (9.8)) and $e^{\mathrm{i}\eta_n''} = 1$ on $Z_n \cap H_r^1$ (cf Lemma 9.3). Hence

$$\partial z_n^+ = \lim_{\substack{v \to v_0 \\ v \in V_n^{mid}}} \partial(\gamma_n e^{\mathrm{i}\eta_n'}).$$

Moreover as in the proof of Lemma 9.2,

$$\gamma_n e^{\mathrm{i}\eta_n'} = \gamma_n\big(-\rho_n + \mathrm{i}\sqrt[+]{1 - \rho_n^2}\big)^{\zeta_n(\lambda_n^-)}$$

where $\rho_n = (\mu_n - \tau_n)/(\gamma_n/2)$. Note that $\rho_n = 0$ on V_n^{mid}. Write $\gamma_n e^{\mathrm{i}\eta_n'}$ as a product

$$\gamma_n e^{\mathrm{i}\eta_n'} = \big(2(\tau_n - \mu_n) + \mathrm{i}\gamma_n \sqrt[+]{1 - \rho_n^2}\big)\big(-\rho_n + \mathrm{i}\sqrt[+]{1 - \rho_n^2}\big)^{\zeta_n(\lambda_n^-) - 1}$$

and note that the first factor converges to 0 as $v \to v_0$ with $v \in V_n^{mid}$ whereas in view of Lemma 9.3 the second factor converges to 1. Furthermore, one verifies using Lemma 9.3 that the gradients of both factors have limits as $v \to v_0$ with $v \in V_n^{mid}$. Therefore

$$\partial z_n^+ = (2\partial\tau_n - 2\partial\mu_n) + \mathrm{i} \lim_{\substack{v \to v_0 \\ v \in V_n^{mid}}} (\partial\lambda_n^+ - \partial\lambda_n^-)$$

where at this point, the limit is to be understood in the sense of H_r^{-1}. We already have discussed ahead of Lemma 9.12 that $\partial\tau_n$ and $\partial\mu_n$ are in L_r^2 and continuous when viewed as maps form H_r^1 to L_r^2. Furthermore, on V_n^{mid}, λ_n^+, λ_n^- are simple eigenvalues and their gradients are given by $\partial\lambda_n^{\pm} = -\frac{\partial\Delta}{\dot\Delta}\big|_{\lambda_n^{\pm}}$. They are also in L_r^2 and continuous when viewed as maps from $H_r^1 \setminus Z_n$ to L_r^2. Note however that $\dot\Delta$ vanishes on Z_n. We now prove that the limits

$$\lim_{\substack{v \to v_0 \\ v \in V_n^{mid}}} \partial\lambda_n^{\pm} = - \lim_{\substack{v \to v_0 \\ v \in V_n^{mid}}} \frac{\partial\Delta}{\dot\Delta}\Big|_{\lambda_n^{\pm}}$$

exist in L_r^2. Recall that $\partial\Delta_\lambda = (f_{\lambda,1}, f_{\lambda,2}P)$ where by (7.10)-(7.11), $f_{\lambda,1}$ and $f_{\lambda,2}$ are linear combinations of products $m_i m_j$ and $e^{\pm q} m_i m_j$ $(1 \le i, j \le 4)$ with coefficients $\grave{m}_2$, $\grave{m}_3$ and δ. Both functions $f_{\lambda,1}$, $f_{\lambda,2}$ are continuous, one periodic and according to Theorem 2.2, in H_r^1 and analytic in v. Note that on $Z_n \cap H_r^1$, $\grave{m}_2$, $\grave{m}_3$, δ, and $\dot\Delta$ all vanish, but $\ddot\Delta(\tau_n) \ne 0$. Hence by de l'Hospital's rule, for $j = 2, 3$,

$$\lim_{\substack{v \to v_0 \\ v \in V_n^{mid}}} \frac{\grave{m}_j}{\dot\Delta}\Big|_{\lambda_n^{\pm}} = \frac{\dot{\grave{m}}_j}{\ddot\Delta}\Big|_{\tau_n}, \quad \lim_{\substack{v \to v_0 \\ v \in V_n^{mid}}} \frac{\delta}{\dot\Delta}\Big|_{\lambda_n^{\pm}} = \frac{\dot\delta}{\ddot\Delta}\Big|_{\tau_n}.$$

Hence in H_r^1 the limits $\lim_{\substack{v \to v_0 \\ v \in V_n^{mid}}} \frac{f_{\lambda_n^{\pm},j}}{\dot\Delta(\lambda_n^{\pm})}$ exist. It then follows that in L_r^2. $\lim_{\substack{v \to v_0 \\ v \in V_n^{mid}}} \partial\lambda_n^{\pm}$ exist. By (7.10)- (7.11) these limits can be explicitly computed. Altogether we have proved the claimed results for z_n^+ with $n \ge 0$. The ones for z_n^- are of course obtained in the same way. The claimed results for z_n^+ and z_n^- with $n \le -1$ are derived in an analogous way. □

Now let us consider $v_0 \in H^1_r \cap Z_n \cap Z_m$. By the same arguments used to show that one can approximate v_0 by potentials in V_n^{mid} one also sees that v_0 can be approximated by potentials in $V_{n,m}^{mid} := V_n^{mid} \cap V_m^{mid}$. Such an approximation is used to compute the Poisson brackets between $z_n^\pm$ and $z_m^\pm$ on $H^1_r \cap Z_n \cap Z_m$.

***Lemma* 9.13** *For any $n, m \in \mathbb{Z}$, the Poisson brackets $\{z_n^\pm, z_m^\pm\}$ are well defined on $H^1_r \cap Z_n \cap Z_m$. If $n \neq m$ they all vanish whereas if $n = m$, one has $\{z_n^+, z_n^-\} = \mathrm{i}\frac{\tau_n}{\ddot{\Delta}(\tau_n)}\dot{\delta}(\tau_n) \neq 0$ (and of course, $\{z_n^+, z_n^+\} = 0$, $\{z_n^-, z_n^-\} = 0$).*

Proof. By Lemma 9.12, for $v_0 \in H^1_r \cap Z_n \cap Z_m$

$$\partial z_n^\pm = \lim_{\substack{v \to v_0 \\ v \in V_{n,m}^{mid}}} (2\partial\tau_n - 2\partial\mu_n) \pm \mathrm{i}(\partial\lambda_n^+ - \partial\lambda_n^-).$$

We recall that $\partial\tau_n$, $\partial\mu_n$, $\partial\lambda_n^\pm$ are continuous when viewed as maps on $V_{n,m}^{mid}$ with values in L^2_r and that the limit is understood to be in L^2_r. In particular $\{z_n^\pm, z_m^\pm\}$ are well defined on $H^1_r \cap Z_n \cap Z_m$. Let us first compute $\{z_n^+, z_m^+\}(v_0)$. It is the limit as $v \to v_0$ with $v \in V_{n,m}^{mid}$ of

$$\{2\tau_n - 2\mu_n + \mathrm{i}(\lambda_n^+ - \lambda_n^-), 2\tau_m - 2\mu_m + \mathrm{i}(\lambda_m^+ - \lambda_m^-)\}.$$

Use that by Theorem 7.4, $\{\lambda_n^\pm, \lambda_m^\pm\} = 0$ and hence also $\{\tau_n, \lambda_m^\pm\} = 0$ to see that the latter bracket equals $-2\mathrm{i}\{\mu_n,\ (\lambda_m^+ - \lambda_m^-)\} - 2\mathrm{i}\{(\lambda_n^+ - \lambda_n^-),\ \mu_m\}$. By Lemma 7.9

$$4\{\mu_n,\ \lambda_m^\pm\} = -\frac{\chi_D(\lambda_m^\pm)}{\dot{\Delta}(\lambda_m^\pm)}\frac{\mu_n}{\dot{\chi}_D(\mu_n)}\frac{\delta(\mu_n)}{\lambda_m^+ - \mu_n}\frac{\lambda_m^+}{\lambda_m^+ + \mu_n}.$$

In case $n \neq m$ we have in the limit $v \to v_0$ by de l'Hospital's rule with $\lambda_n^+ = \lambda_n^- = \tau_n$, $\lambda_m^+ = \lambda_m^- = \tau_m$

$$-\frac{\dot{\chi}_D(\tau_m)}{\ddot{\Delta}(\tau_m)}\frac{\tau_n}{\dot{\chi}_D(\tau_n)}\frac{\tau_m}{\tau_m + \tau_n}\frac{\delta(\tau_n)}{\tau_m - \tau_n} = 0$$

since all factors are non zero except $\delta(\tau_n) = 0$. It implies that for $n \neq m$, $\{z_n^+, z_m^+\} = 0$. Similarly one shows that $\{z_n^\pm, z_m^\pm\} = 0$ for any choices of + and -. In casse $n = m$, clearly $\{z_n^+, z_n^+\} = 0$ and $\{z_n^-, z_n^-\} = 0$ by the skew symmetry of the Poisson bracket. For the approximating brackets of $\{z_n^+, z_n^-\}$ we obtain $-4\mathrm{i}\{\mu_n, \lambda_n^+ - \lambda_n^-\}$ which again by Lemma 7.9 equals

$$\frac{\mathrm{i}\chi_D(\lambda_n^+)}{\dot{\Delta}(\lambda_n^+)}\frac{\mu_n}{\dot{\chi}_D(\mu_n)}\frac{\lambda_n^+}{\lambda_n^+ + \mu_n}\frac{\delta(\mu_n)}{\lambda_n^+ - \mu_n} - \frac{\mathrm{i}\chi_D(\lambda_n^-)}{\dot{\Delta}(\lambda_n^-)}\frac{\mu_n}{\dot{\chi}_D(\mu_n)}\frac{\lambda_n^+}{\lambda_n^- + \mu_n}\frac{\delta(\mu_n)}{\lambda_n^- - \mu_n}.$$

Since for $v \in V_n^{mid}$, $\mu_n = \tau_n$, $\lambda_n^\pm = \tau_n \pm \gamma_n/2$ and the expression of δ at λ_n^- takes the form

$$\delta(\tau_n) = 0 + \dot{\delta}(\lambda_n^-)\gamma_n/2 + O(\gamma_n^2)$$

the limit of each of the two terms can be computed individually. These limits are equal and in all one gets in the limit $v \to v_0$

$$2\mathrm{i}\frac{\dot{\chi}_D(\tau_n)}{\ddot{\Delta}(\tau_n)}\frac{\tau_n}{\dot{\chi}_D(\tau_n)}\frac{\tau_n}{2\tau_n}\dot{\delta}(\tau_n) = \mathrm{i}\frac{\tau_n}{\ddot{\Delta}(\tau_n)}\dot{\delta}(\tau_n).$$

Since $\delta(\mu_n)^2 = \Delta^2(\mu_n) - 1$ and by the product representation of $\Delta^2(\mu_n) - 1$, with $\mu_n = \tau_n$,

$$\frac{\delta(\mu_n)^2}{(\gamma_n/2)^2} = \frac{\Delta^2(\mu_n) - 1}{(\gamma_n/2)^2}$$

is locally uniformly bounded away from zero on H^1_r, it follows that $\dot{\delta}(\tau_n) \neq 0$. □

***Proposition* 9.14** *For any $n, m \in \mathbb{Z}$, the Poisson brackets $\{x_n, x_m\}$, $\{x_n, y_m\}$, $\{y_n, x_m\}$, and $\{y_n, y_m\}$ are well defined on $H^1_r \cap Z_n \cap Z_m$. If $n \neq m$ they all vanish whereas if $n = m$ one has $\{x_n, y_n\} = \mathrm{i}\frac{\xi_n^2}{\tau_n}\{z_n^+, z_n^-\} \neq 0$ (and of course, $\{x_n, x_n\} = 0$, $\{y_n, y_n\} = 0$).*

Proof. Recall from Section 9.1 that on H^1_r,

$$x_n = \frac{\xi_n}{\sqrt[+]{2\tau_n}}(e^{\mathrm{i}\beta_n} z_n^+ + e^{-\mathrm{i}\beta_n} z_n^-),$$
$$y_n = \frac{\xi_n}{\mathrm{i}\sqrt[+]{2\tau_n}}(e^{\mathrm{i}\beta_n} z_n^+ - e^{-\mathrm{i}\beta_n} z_n^-).$$

Since $z_n^\pm$ vanish on Z_n it follows that on Z_n

$$\partial x_n = \frac{\xi_n}{\sqrt[+]{2\tau_n}}(e^{\mathrm{i}\beta_n}\partial z_n^+ + e^{-\mathrm{i}\beta_n}\partial z_n^-)$$

and

$$\partial y_n = \frac{\xi_n}{\mathrm{i}\sqrt[+]{2\tau_n}}(e^{\mathrm{i}\beta_n}\partial z_n^+ - e^{-\mathrm{i}\beta_n}\partial z_n^-).$$

By Lemma 9.13 one then concludes that on $Z_n \cap Z_m$ the Poisson brackets $\{x_n, x_m\}$, $\{x_n, y_m\}$, $\{y_n, x_m\}$, and $\{y_n, y_m\}$ are well defined and in case $n \neq m$ all vanish. In case $n = m$, clearly $\{x_n, x_n\} = 0$ and $\{y_n, y_n\} = 0$ by the skew symmetry of the Poisson bracket whereas

$$\{x_n, y_n\} = \frac{\xi_n^2}{\mathrm{i}2\tau_n}\{e^{\mathrm{i}\beta_n}z_n^+ + e^{-\mathrm{i}\beta_n}z_n^-,\ e^{\mathrm{i}\beta_n}z_n^+ - e^{-\mathrm{i}\beta_n}z_n^-\} = \frac{\xi_n^2}{\mathrm{i}2\tau_n}2\{z_n^-,\ z_n^+\} = \mathrm{i}\frac{\xi_n^2}{\tau_n}\{z_n^+,\ z_n^-\} \neq 0.$$

□

9.3 Differential of the Birkhoff map

In this section we study the differential $\mathrm{d}_v\Phi$ of Φ for $v \in H_r^1$. Since by Theorem 9.6, $\Phi : W \to h_c^{1/2}$ is real analytic, its Jacobian

$$\mathrm{d}\Phi : W \to \mathcal{L}(H_c^1, h_c^{1/2}),\ v \mapsto \mathrm{d}_v\Phi$$

is a real analytic map. Here $\mathcal{L}(H_c^1, h_c^{1/2})$ denotes the Banach space of bounded linear operators from H_c^1 to $h_c^{1/2}$, endowed with the operator nom. As a first result we obtain a formula for $\mathrm{d}_v\Phi$ at $v = 0$ and show that it is a linear isomorphism. We begin by deriving formulas for the gradients ∂z_n^+, ∂z_n^- at a potential $v \in Z_n \cap H_r^1$ with $n \geq 0$. Let

$$\phi_n(\lambda) := \frac{\zeta_n(\mu_n)}{\psi_n(\mu_n)}, \quad \zeta_n(\lambda) = \Big(\prod_{k\neq n}\frac{\sigma_{1,k}^n - \lambda}{w_{1,k}(\lambda)}\Big)\frac{\psi_{n,2}(\lambda)/\psi_{n,2}(\infty)}{\sqrt[c]{\chi_2(\lambda)}/\sqrt[c]{\chi_2(\infty)}} \tag{9.19}$$

***Lemma* 9.15** *For any $v \in Z_n \cap H_r^1$ with $n \geq 0$,*

$$\partial(z_n^\pm) = 2(\partial\tau_n - \partial\mu_n) \pm 2\mathrm{i}\big(\phi_n\partial\delta\big|_{\mu_n} + \phi_n\dot\delta(\mu_n)\partial\mu_n\big).$$

Proof. Let $v_0 \in Z_n \cap H_r^1$ with $n \geq 0$. By Proposition 6.10 and Corollary 7.15, v_0 can be approximated by potentials in $v \in H_r^1 \setminus Z_n$ with $\lambda_n^- < \mu_n < \lambda_n^+$. For such a potential one has

$$\mathrm{i}\eta_n = \mathrm{i}\int_{\lambda_n^-}^{\mu_n}\frac{\psi_n(\lambda)}{\sqrt[*]{\chi_p(\lambda)}}\,\mathrm{d}\lambda = -\varepsilon_n\int_{\lambda_n^-}^{\mu_n}\frac{\zeta_n(\lambda)}{w_{1,n}}\,\mathrm{d}\lambda$$

where $\varepsilon_n = \frac{\sqrt[*]{\chi_p(\mu_n)}}{\sqrt[c]{\chi_p(\mu_n)}}$ and $w_{1,n}(\lambda)$ denotes the standard root $w_{1,n}(\lambda) = \sqrt[s]{(\lambda_n^+ - \lambda)(\lambda_n^- - \lambda)}$ which we extend to G_n by setting $w_{1,n}(\lambda) = \lim_{\substack{\epsilon\to 0\\ \epsilon>0}} w_{1,n}(\lambda + \mathrm{i}\epsilon)$, meaning that $w_{1,n}(\lambda) = -\mathrm{i}\sqrt[+]{(\lambda_n^+ - \lambda)(\lambda - \lambda_n^-)}$ for any $\lambda_n^- \leq \lambda \leq \lambda_n^+$.

We follow the proof of Lemma 9.2 and use the notation introduced there. By (9.7) one obtains (with $\rho_n = 2(\mu_n - \tau_n)/\gamma_n$)

$$\gamma_n e^{-\varepsilon_n\omega_n} = -\gamma_n\rho_n - \mathrm{i}\varepsilon_n\gamma_n\sqrt[+]{(1-\rho_n)(\rho_n+1)} = 2(\tau_n - \mu_n) - \mathrm{i}2\varepsilon_n\sqrt[+]{(\lambda_n^+ - \mu_n)(\mu_n - \lambda_n^-)}.$$

Furthermore, since both $\gamma_n e^{-\varepsilon_n\omega_n}$ and $e^{-\varepsilon_n(\upsilon_n + o_n)}$ are analytic and since $\lim_{v\to v_0}(\upsilon_n + o_n) = 0$, the product rule yields

$$\partial z_n^+ = \partial(\gamma_n e^{\mathrm{i}\eta_n}) \to \partial(\gamma_n e^{-\varepsilon_n\omega_n}) = 2\partial(\tau_n - \mu_n) - \mathrm{i}2\varepsilon_n\partial r_n$$

where we set

$$r_n = \sqrt[+]{(\lambda_n^+ - \mu_n)(\mu_n - \lambda_n^-)}.$$

To study the gradient of r_n we use that

$$\frac{\psi_n(\lambda)}{\sqrt[*]{\chi_p(\lambda)}} = \mathrm{i}\varepsilon_n \frac{\zeta_n(\lambda)}{w_{1,n}(\lambda)}$$

and $w_{1,n}(\mu_n) = -\mathrm{i}r_n$ to conclude that

$$\varepsilon_n r_n = -\frac{\zeta_n(\mu_n)}{\psi_n(\mu_n)} \sqrt[*]{\chi_p(\mu_n)} = -\phi_n \delta(\mu_n),$$

where we recall that by the definition (9.19), $\phi_n = \zeta_n(\mu_n)/\psi_n(\mu_n)$. Since $v \in Z_n \cap H^1_r$, $\delta(\mu_n) = 0$ and hence

$$-\varepsilon_n \partial r_n = \phi_n \partial\, \delta\big|_{\mu_n} + \phi_n \dot{\delta}(\mu_n)\partial\mu_n$$

implying that

$$\partial z_n^+ = 2\partial(\tau_n - \mu_n) + 2\mathrm{i}\big(\phi_n \partial\, \delta\big|_{\mu_n} + \phi_n \dot{\delta}(\mu_n)\partial\mu_n\big).$$

Going through the arguments of the proof on obtains the formula

$$\partial z_n^- = 2\partial(\tau_n - \mu_n) - 2\mathrm{i}\big(\phi_n \partial\, \delta\big|_{\mu_n} + \phi_n \dot{\delta}(\mu_n)\partial\mu_n\big).$$

□

In the case where $v = 0$, the formulas of Lemma 9.15 for $\partial z_n^\pm$ can be explicitly computed. By Corollary 3.10, $\lambda_n^-(0) = \lambda_n^+(0) = \mu_n(0) = \tau_n(0)$ and

$$\mu_n(0) = \frac{n\pi}{2} + \frac{1}{4}\langle 2n\rangle \quad \forall\ n \geq 0, \tag{9.20}$$

where we recall that $\langle n\rangle = (n^2\pi^2 + 1)^{1/2}$. One has

$$\frac{1}{16\mu_n(0)} = -\frac{n\pi}{2} + \frac{1}{4}\langle 2n\rangle, \quad \mu_n(0) + \frac{1}{16\mu_n(0)} = \frac{1}{2}\langle 2n\rangle. \tag{9.21}$$

***Lemma* 9.16** *For $v = 0$ and $n \geq 0$,*

$$\partial z_n^+(0) = \tau_n(0)(-e^{\mathrm{i}2\pi nx},\ -\mathrm{i}e^{\mathrm{i}2\pi nx}), \quad \partial z_n^-(0) = \tau_n(0)(-e^{-\mathrm{i}2\pi nx},\ \mathrm{i}e^{-\mathrm{i}2\pi nx}).$$

Proof. Let $n \geq 0$. We need to compute the terms appearing in the formulas for $\partial z_n^\pm$ of Lemma 9.15. To ease notation we write $\partial z_n^\pm$ instead of $\partial z_n^\pm(0)$ and similarly for other quantities.

Recall that the fundamental solution at $v = 0$ is given by

$$M(x,\lambda) = E_{\omega(\lambda)}(x) = \begin{pmatrix} \cos(\omega(\lambda)x) & \sin(\omega(\lambda)x) \\ -\sin(\omega(\lambda)x) & \cos(\omega(\lambda)x) \end{pmatrix}$$

where $\omega(\lambda) = \lambda - (16\lambda)^{-1}$. In particular, $\omega(\mu_n) = n\pi$. It then follows that $\delta(\lambda) = 0$, hence $\dot{\delta}(\lambda) = 0$, and $\Delta(\lambda) = \cos(\omega(\lambda))$ for any $\lambda \in \mathbb{C}^*$. By Lemma 5.2, one has for any $\lambda \in \mathbb{C}^*$, $\partial\Delta_\lambda = 0$,

$$\partial_q \delta(\lambda) = \frac{1}{2}\Big(\lambda + \frac{1}{16\,\lambda}\Big)\Big(\cos(\omega(\lambda))\sin(2\omega(\lambda)x) - \sin(\omega(\lambda))\cos(2\omega(\lambda)x)\Big)$$

and

$$\partial_p \delta(\lambda) = \frac{1}{4}\Big(\cos(\omega(\lambda))\cos(2\omega(\lambda)x) + \frac{1}{4}\sin(\omega(\lambda))\sin(2\omega(\lambda)x)\Big)P(\cdot).$$

Furthermore by (6.11)

$$2\partial\tau_n = \frac{1}{2\pi\mathrm{i}}\int_{\partial U_n} \partial_\lambda\left(\frac{\Delta(\lambda)\partial\Delta(\lambda)}{\chi_p(\lambda)}\right)\lambda\,\mathrm{d}\lambda = 0$$

and

$$\begin{aligned} \partial_q\delta\big|_{\lambda=\mu_n} &= \frac{(-1)^n}{2}\Big(\mu_n + \frac{1}{16\,\mu_n}\Big)\sin(2\pi nx), \\ \partial_p\delta\big|_{\lambda=\mu_n} &= \frac{(-1)^n}{4}P(\cos(2\pi nx)) = \frac{(-1)^n}{4}\langle 2n\rangle\cos(2\pi nx). \end{aligned}$$

When combined with (9.21) one gets

$$\partial\delta\big|_{\lambda=\mu_n} = \frac{(-1)^n}{4}\langle 2n\rangle\big(\sin(2\pi nx),\ \cos(2\pi nx)\big).$$

By the definition of ζ_n, one has

$$\frac{\psi_n}{\sqrt[*]{\chi_p}} = \mathrm{i}\varepsilon_n\frac{\zeta_n}{w_{1,n}}$$

and hence

$$\frac{\zeta_n(\lambda)}{\psi_n(\lambda)} = -\mathrm{i}\frac{w_{1,n}(\lambda)}{\varepsilon_n\sqrt[*]{\chi_p(\lambda)}} = \frac{1}{\mathrm{i}}\frac{w_{1,n}(\lambda)}{\sqrt[c]{\chi_p(\lambda)}}. \tag{9.22}$$

Since by Lemma 6.21, $\sqrt[c]{\chi_p(\lambda)} = -\mathrm{i}\sin(\omega(\lambda))$ and by the definition of the standard root, $w_{1,n}(\lambda) = \mu_n - \lambda$ for $v = 0$, one sees that for $v = 0$

$$\phi_n = \frac{\zeta_n(\mu_n)}{\psi_n(\mu_n)} = \lim_{\lambda\to\mu_n}\frac{(\mu_n-\lambda)}{\sin(\omega(\lambda))} = \lim_{\lambda\to\mu_n}\frac{-1}{\cos(\omega(\lambda))\dot\omega(\lambda)} = (-1)^{n+1}\frac{1}{1+\frac{1}{16\,\mu_n^2}} = (-1)^{n+1}\frac{\mu_n}{\mu_n+\frac{1}{16\,\mu_n}}.$$

Altogether we have proved that for $v = 0$, $\phi_n = (-1)^{n+1}\frac{\mu_n}{\frac{1}{2}\langle 2n\rangle}$ (cf (9.21)) yielding

$$2\mathrm{i}\phi_n\partial\delta\big|_{\mu_n} = -\mathrm{i}\mu_n(\sin(2\pi nx), \cos(2\pi nx)).$$

It remains to compute $\partial\mu_n$ in the formulas for $\partial z_n^{\pm}$ of Lemma 9.15. By Lemma 5.6,

$$\partial\mu_n = \frac{\grave{m}_1}{\dot\chi_D}\Big|_{\mu_n}[\![M_2]\!]_{0,\mu_n}.$$

Since $\grave{m}_1(\mu_n) = \cos(\omega(\mu_n))$ and $\chi_D(\lambda) = \sin(\omega(\lambda))$ one has

$$\frac{\grave{m}_1}{\dot\chi_D}\Big|_{\mu_n} = \frac{1}{\dot\omega(\mu_n)} = (1+\frac{1}{16\mu_n^2})^{-1} = \frac{\mu_n}{\frac{1}{2}\langle 2n\rangle}.$$

Furthermore, by (5.10) and the definition of the $\star$ product $\begin{pmatrix}a_1\\a_2\end{pmatrix}\star\begin{pmatrix}b_1\\b_2\end{pmatrix} = a_1b_1 - a_2b_2$ one gets

$$\begin{aligned}[\![M_2]\!]_{0,\mu_n} =& \frac{1}{2}\begin{pmatrix}-(\mu_n+\frac{1}{16\mu_n})M_2\star M_2\\ -Pm_2m_4\end{pmatrix}\Big|_{\mu_n} = \frac{1}{2}\begin{pmatrix}\frac{1}{2}\langle 2n\rangle\cos(2\pi nx)\\ -\frac{1}{2}\langle 2n\rangle\sin(2\pi nx)\end{pmatrix}\\ =&\frac{1}{4}\langle 2n\rangle\begin{pmatrix}\cos(2n\pi x)\\ -\sin(2\pi nx)\end{pmatrix}.\end{aligned}$$

Altogether we have proved that

$$\partial\mu_n = \frac{\mu_n}{2}(\cos(2\pi nx),\ -\sin(2\pi nx)).$$

Substituting the formulas obtained into the expression for $\partial z_n^{\pm}$ of Lemma 9.15 we conclude, (since $\mu_n = \tau_n$ for $v = 0$)

$$\partial(z_n^+) = -\tau_n\begin{pmatrix}\cos(2\pi nx)\\ -\sin(2\pi nx)\end{pmatrix} - \mathrm{i}\tau_n\begin{pmatrix}\sin(2\pi nx)\\ \cos(2\pi nx)\end{pmatrix} = \tau_n(-e^{\mathrm{i}2n\pi x},\ -\mathrm{i}e^{\mathrm{i}2\pi nx}).$$

In the same way one gets

$$\partial z_n^- = \tau_n(-e^{-\mathrm{i}2\pi nx},\ \mathrm{i}e^{-\mathrm{i}2\pi nx})$$

□

As an application of Lemma 9.16 we obtain an explicit formula for $\mathrm{d}_0\Phi$. Let for any $n\in\mathbb{Z}$

$$d_n^+(x) := \begin{pmatrix}-\cos(2\pi nx)\\ \sin(2\pi nx)\end{pmatrix},\quad d_n^- := \begin{pmatrix}-\sin(2\pi nx)\\ -\cos(2\pi nx)\end{pmatrix}.$$

Note that d_n^+, d_n^-, $n\in\mathbb{Z}$ form an orthonormal basis of L_r^2 (and L_c^2).

***Proposition* 9.17** *(i) For any $n \in \mathbb{Z}$*

$$\partial x_n(0) = \langle 2n\rangle^{1/2} d_n^+, \qquad d_n^+ = \frac{1}{2\tau_n(0)}(\partial z_n^+(0) + \partial z_n^-(0))$$
$$\partial y_n(0) = \langle 2n\rangle^{1/2} d_n^-, \qquad d_n^- = \frac{1}{2\tau_n(0)}\frac{1}{\mathrm{i}}(\partial z_n^+(0) - \partial z_n^-(0))$$

(ii) The Jacobian of Φ at $v = 0$ is given by the weighted Fourier transform

$$\mathcal{F} : H_c^1 \to h_c^{1/2},\ \mathring{v} \mapsto \big((\langle 2n\rangle^{1/2}\langle \mathring{v}, d_n^+\rangle_r)_{n\in\mathbb{Z}}, (\langle 2n\rangle^{1/2}\langle \mathring{v}, d_n^-\rangle_r)_{n\in\mathbb{Z}}\big).$$

Hence $\mathrm{d}_0\Phi$ is a linear isomorphism and its inverse is given by

$$\mathcal{F}^{-1} : h_c^{1/2} \to H_c^1,\ ((\mathring{x}_n)_n,\ (\mathring{y}_n)_n) \mapsto \sum_{n\in\mathbb{Z}}\Big(\frac{\mathring{x}_n}{\langle 2n\rangle^{1/2}} d_n^+ + \frac{\mathring{y}_n}{\langle 2n\rangle^{1/2}} d_n^-\Big).$$

More explicitely, $(\mathring{q}, \mathring{p}) := \mathcal{F}^{-1}((\mathring{x}_n)_n,\ (\mathring{y}_n)_n)$ is given by

$$\mathring{q}(x) = -\,\mathring{x}_0 - \sum_{n\geq 1}\frac{\mathring{x}_n + \mathring{x}_{-n}}{\langle 2n\rangle^{1/2}}\cos(2\pi n x) - \sum_{n\geq 1}\frac{\mathring{y}_n - \mathring{y}_{-n}}{\langle 2n\rangle^{1/2}}\sin(2\pi n x)$$
$$\mathring{p}(x) = -\,\mathring{y}_0 - \sum_{n\geq 1}\frac{\mathring{y}_n + \mathring{y}_{-n}}{\langle 2n\rangle^{1/2}}\cos(2\pi n x) + \sum_{n\geq 1}\frac{\mathring{x}_n - \mathring{x}_{-n}}{\langle 2n\rangle^{1/2}}\sin(2\pi n x).$$

Proof. Let $n \geq 0$. For ease of notation we again write $z_n^\pm$ instead of $z_n^\pm(0)$ and similarly for other quantities. By (9.12) and since $z_n^\pm = 0$ one has

$$\partial x_n = \frac{\xi_n}{\sqrt[+]{2\tau_n}}\left(e^{\mathrm{i}\beta_n}\partial z_n^+ + e^{-\mathrm{i}\beta_n}\partial z_n^-\right), \qquad \partial y_n = \frac{\xi_n}{\mathrm{i}\sqrt[+]{2\tau_n}}\left(e^{\mathrm{i}\beta_n}\partial z_n^+ - e^{-\mathrm{i}\beta_n}\partial z_n^-\right).$$

By the definition of β_n, $\beta_n = 0$ and by Theorem 8.8,

$$\xi_n = \sqrt[+]{\frac{1}{2\tau_n}\sqrt[+]{4\pi^2 n^2 + 1}} = \frac{\langle 2n\rangle^{1/2}}{\sqrt[+]{2\tau_n}}$$

where $\tau_n = \lambda_n^+ (= \lambda_n^-) = \mu_n$. Hence by Lemma 9.16 for $n \geq 0$

$$\partial x_n = \frac{\langle 2n\rangle^{1/2}}{2\tau_n}(\partial z_n^+ + \partial z_n^-) = \langle 2n\rangle^{1/2} d_n^+ \tag{9.23}$$

and

$$\partial y_n = \frac{\langle 2n\rangle^{1/2}}{\mathrm{i}2\tau_n}(\partial z_n^+ - \partial z_n^-) = \langle 2n\rangle^{1/2} d_n^-. \tag{9.24}$$

For $n \geq 1$, we have for any $(q,p) \in H_r^1$, $x_{-n}(q,p) = -x_n(-q,p)$ and $y_{-n}(q,p) = y_n(-q,p)$. Hence

$$(\partial_q x_{-n},\ \partial_p x_{-n}) = (\partial_q x_n,\ -\partial_p x_n)\big|_{-q,p}$$

and

$$(\partial_q y_{-n},\ \partial_p y_{-n}) = (-\partial_q y_n,\ \partial_p y_n)\big|_{-q,p}.$$

For $(q,p) = (0,0)$ it then follows from (9.23)-(9.24) that for $n \geq 1$,

$$\partial x_{-n} = \langle 2n\rangle^{1/2}(-\cos(2\pi n x),\ -\sin(2\pi n x)) = \langle 2n\rangle^{1/2} d_{-n}^+$$
$$\partial y_{-n} = \langle 2n\rangle^{1/2}(\sin(2\pi n x),\ -\cos(2\pi n x)) = \langle 2n\rangle^{1/2} d_{-n}^-.$$

This proves (i). Item (ii) follows from (i) in a straightforward way. □

***Lemma* 9.18** *At a right sided finite-gap potential v in H_r^1, $\partial z_n^\pm(v)$ admits for $n \to \infty$ the asymptotics*

$$\partial z_n^\pm(v) = \partial z_n^\pm(0) + (\ell_n^2 \partial_x(\cdot) + \ell_n^2,\ \ell_n^2 P(\cdot)).$$

Proof. Let $v \in H^1_r$ be a right sided finite gap potential and set $\mathcal{S}_+ := \{ n \geq 0 : \gamma_n > 0 \}$. Then $\mathcal{S}_+$ is finite. By Lemma 9.15, for any $n \geq 0$ with $n \notin \mathcal{S}_+$, $\partial z_n^\pm \equiv \partial z_n^\pm(v)$ is given by

$$\partial(z_n^\pm) = 2(\partial \tau_n - \partial \mu_n) \pm 2\mathrm{i}\big(\phi_n(\mu_n)\partial\, \delta|_{\mu_n} + \phi_n \dot{\delta}(\mu_n)\partial\mu_n\big).$$

Let us discuss the terms of this formula in more detail. By Lemma 6.9, $\grave{M}(\mu_n) = (-1)^n I$, implying that $\delta(\mu_n) = 0$ and $\Delta(\mu_n) = (-1)^n$. Since by Lemma 3.16, $\mu_n = n\pi + \ell_n^2$ as $n \to \infty$, one has by Corollary 5.16

$$\partial\, \delta|_{\mu_n} = \frac{(-1)^n}{4}\,(\cos(2\pi nx)\cdot \partial_x(\cdot),\ \cos(2\pi nx)P(\cdot)) + (\ell_n^2 \cdot \partial_x(\cdot) + \ell_n^2,\ \ell_n^2 \cdot P(\cdot)).$$

By Lemma 6.8, as $n \to \infty$,

$$\partial \tau_n = (\ell_n^2 \cdot \partial_x(\cdot) + \ell_n^2,\ \ell_n^2 \cdot P(\cdot))$$

and, by Theorem 2.12(iv) for $n \to \infty$

$$M(x,\mu_n) = E_{n\pi}(x) + \ell_n^2, \qquad \dot{M}(x,\mu_n) = xJE_{n\pi}(x) + \ell_n^2,$$

where we recall that by (2.31)

$$E_{n\pi}(x) = \begin{pmatrix} \cos(\pi nx) & \sin(\pi nx) \\ -\sin(n\pi x) & \cos(\pi nx) \end{pmatrix}.$$

Hence $\dot{\delta}(\mu_n) = \ell_n^2$. Furthermore by Lemma 5.17

$$\partial\mu_n = -\frac{1}{4}\Big(\sin(2\pi nx)\cdot\partial_x(\cdot),\ \sin(2\pi nx)\cdot P(\cdot)\Big) + (\ell_n^2 \cdot \partial_x(\cdot) + \ell_n^2,\ \ell_n^2 P(\cdot)).$$

Combining these estimates yields

$$\begin{aligned}\partial(z_n^\pm) =& \frac{1}{2}\Big(\sin(2\pi nx)\cdot\partial_x(\cdot),\ \sin(2\pi nx)\cdot P(\cdot)\Big) \\ &\pm \mathrm{i}\frac{(-1)^n}{2}\phi_n\Big(\cos(2\pi nx)\cdot\partial_x(\cdot),\ \cos(2\pi nx)\cdot P(\cdot)\Big) \\ &+ \phi_n\left(\ell_n^2\cdot\partial_x(\cdot) + \ell_n^2,\ \ell_n^2\cdot P(\cdot)\right) + \left(\ell_n^2\cdot\partial_x(\cdot) + \ell_n^2,\ \ell_n^2\cdot P(\cdot)\right).\end{aligned}$$

It remains to compute the asymptotics of ϕ_n. Since $\frac{\zeta_n(\lambda)}{\psi_n(\lambda)} = -\mathrm{i}\frac{w_{1,n}(\lambda)}{\sqrt[c]{\chi_p(\lambda)}}$ one gets by Lemma 6.23(i)

$$\phi_n = -\mathrm{i}\frac{w_{1,n}(\mu_n)}{\sqrt[c]{\chi_p(\mu_n)}} = (-1)^{n+1} + \ell_n^2 \quad \text{as } n \to \infty.$$

Hence we have proved that

$$\begin{aligned}\partial z_n^\pm =& \frac{1}{2}(\sin(2\pi nx)\cdot\partial_x(\cdot),\ \sin(2\pi nx)\cdot P(\cdot)) \\ &- \pm\frac{\mathrm{i}}{2}(\cos(2\pi nx)\cdot\partial_x(\cdot),\ \cos(2\pi nx)\cdot P(\cdot)) + (\ell_n^2\cdot\partial_x(\cdot) + \ell_n^2,\ \ell_n^2 P(\cdot)).\end{aligned}$$

Integrating by parts when evaluating the leading order part of $\mathrm{d}z_n^\pm[\mathring{v}]$ and using that P is selfadjoint and $P(e^{\mathrm{i}2\pi nx}) = \langle 2n\rangle e^{\mathrm{i}2\pi nx}$ one gets $(-n\pi e^{\pm\mathrm{i}2\pi nx},\ -\pm\mathrm{i}\frac{\langle 2n\rangle}{2}e^{\pm\mathrm{i}2\pi nx})$.

Since $\mu_n(0) = \frac{n\pi}{2} + \frac{1}{4}\langle 2n\rangle = n\pi + O(\frac{1}{n})$ and $\frac{\langle 2n\rangle}{2} = n\pi + O(\frac{1}{n})$ we conclude from Lemma 9.16 that as $n \to \infty$

$$\partial z_n^\pm(v) = \partial z_n^\pm(0) + (\ell_n^2\cdot\partial_x(\cdot),\ \ell_n^2 P(\cdot)).$$

□

Next we want to obtain asymptotic estimates for $\mathrm{d}_v\,\Phi$ in the case where $v \in H^1_r$ is a right sided finite gap potential. For any $n \in \mathbb{Z}$, let

$$b_n^+ \equiv b_n^+(v) := \langle 2n\rangle^{-1/2}\partial x_n, \quad b_n^- \equiv b_n^-(v) := \langle 2n\rangle^{-1/2}\partial y_n.$$

One then has for any $\mathring{v} \in H^1_r$

$$\mathrm{d}x_n[\mathring{v}] = \langle 2n\rangle^{-1/2}\langle \mathring{v}, b_n^+\rangle, \quad \mathrm{d}y_n[\mathring{v}] = \langle 2n\rangle^{-1/2}\langle \mathring{v}, b_n^-\rangle$$

and hence $\mathrm{d}\Phi[\mathring{v}] = \big((\langle 2n\rangle^{1/2}\langle \mathring{v},\ b_n^+\rangle)_n,\ (\langle 2n\rangle^{1/2}\langle \mathring{v}, b_n^-\rangle)_n\big)$.

***Lemma* 9.19** *For any right sided finite gap potential $v \in H^1_r$, one has for $n \to \infty$*

$$b_n^\pm = d_n^\pm + \langle n\rangle^{-1}(\ell_n^2 \partial_x(\cdot) + \ell_n^2,\ \ell_n^2 P(\cdot)).$$

Expressed in more detail, it means that for any $\mathring{v} = (\mathring{q}, \mathring{p}) \in H^1_r$

$$\langle \mathring{v}, b_n^\pm\rangle = \langle \mathring{v}, d_n^\pm\rangle + \langle n\rangle^{-1}(\langle \partial_x \mathring{q}, \ell_n^2\rangle + \langle \mathring{q}, \ell_n^2\rangle + \langle P\mathring{p}, \ell_n^2\rangle_r).$$

For the left sided finite gap potentials, the latter asymptotic estimates hold as $n \to -\infty$.

Proof. Let $v \in H^1_r$ be a right sided finite-gap potential and set $\mathcal{S}_+ := \{\ n \geq 0 :\ \gamma_n > 0\ \}$. Then $\mathcal{S}_+$ is finite. By (9.12) and since $z_n^\pm \equiv z_n^\pm(v) = 0$ for $n \geq 0$ with $n \notin \mathcal{S}_+$, one has for any such n

$$\partial x_n = \frac{\xi_n}{\sqrt[+]{2\tau_n}}\left(e^{\mathrm{i}\beta_n}\partial z_n^+ + e^{-\mathrm{i}\beta_n}\partial z_n^-\right), \qquad \partial y_n = \frac{\xi_n}{\mathrm{i}\sqrt[+]{2\tau_n}}\left(e^{\mathrm{i}\beta_n}\partial z_n^+ - e^{-\mathrm{i}\beta_n}\partial z_n^-\right).$$

Recall that $\tau_n = \mu_n = \pi_n + \ell_n^2$ (cf Lemma 3.17), $\xi_n = 1 + \ell_n^2$ (Theorem 8.8) and $\beta_n = \sum_{k\neq n}\beta_{1,k}^n + \sum_{k\in\mathbb{Z}}\beta_{2,k}^n$ (cf (9.4)), which satisfies $\sum_{k\in\mathbb{Z}}\beta_{2,k}^n = O(\frac{1}{n})$ (Theorem 8.31) and $\sum_{k\neq n}\beta_{1,k}^n = O(\frac{1}{n})$ (Lemma 8.28 for v right sided finite gap potential). Hence for $n \to \infty$

$$\partial x_n = \langle 2n\rangle^{-1/2}(1+\ell_n^2)\partial z_n^+ + \langle 2n\rangle^{-1/2}(1+\ell_n^2)\partial z_n^-$$

and

$$\partial y_n = \langle 2n\rangle^{-1/2}(1+\ell_n^2)\frac{1}{\mathrm{i}}\partial z_n^+ - \langle 2n\rangle^{-1/2}(1+\ell_n^2)\frac{1}{\mathrm{i}}\partial z_n^-.$$

By Lemma 9.18, the leading order term of $\partial z_n^\pm(v)$ is $\partial z_n^\pm(0)$. Using that by Proposition 9.17(i),

$$\partial z_n^+(0) + \partial z_n^-(0) = 2\tau_n(0)d_n^+, \quad \frac{1}{\mathrm{i}}(\partial z_n^+(0) - \partial z_n^-(0)) = 2\tau_n(0)d_n^-$$

it follows that the leading order term of ∂x_n is $\langle 2n\rangle^{-1/2}2\tau_n(0)d_n^+$ whereas the one of ∂y_n is $\langle 2n\rangle^{-1/2}2\tau_n(0)d_n^-$. Since $\langle 2n\rangle^{-1/2}2\tau_n(0) = \langle 2n\rangle^{1/2} + \frac{1}{\langle n\rangle^{1/2}}\ell_n^2$ and taking into account the error terms for the asymptotic estimates of $\partial z_n^\pm$ of Lemma 9.18 it follows that

$$\partial x_n = \langle 2n\rangle^{1/2}d_n^+ + \langle n\rangle^{-1/2}(\ell_n^2\partial_x(\cdot) + \ell_n^2,\ \ell_n^2 P(\cdot))$$

and

$$\partial y_n = \langle 2n\rangle^{1/2}d_n^- + \langle n\rangle^{-1/2}(\ell_n^2\partial_x(\cdot) + \ell_n^2,\ \ell_n^2 P(\cdot)).$$

Since by definition, $\partial x_n = \langle 2n\rangle^{1/2}b_n^+$ and $\partial y_n = \langle 2n\rangle^{1/2}b_n^-$, the claimed asymptotics for right sided finite gap potentials are proved. In case $v = (q,p)$ is a left sided finite gap potential, Lemma 6.6(ii) says that $(-q,p)$ is a right sided finite gap potential. Since for $n \geq 1$, $x_{-n} = -x_n(-q,p)$ and $y_{-n} = y_n(-q,p)$ and hence

$$\partial x_{-n} = (\partial_q x_{-n}, \partial_p x_{-n}) = (\partial_q x_n\big|_{(-q,p)},\ -\partial_p x_n\big|_{(-q,p)})$$
$$\partial y_{-n} = (\partial_q y_{-n}, \partial_p y_{-n}) = (-\partial_q y_n\big|_{(-q,p)},\ \partial_p y_n\big|_{(-q,p)})$$

the claimed asymptotic estimates follow from the ones of ∂x_n, ∂y_n at $(-q,p)$ for $n \to \infty$. □

The next result says that for any $v \in H^1_r$, $\mathrm{d}_v\,\Phi$ is a compact perturbation of the weighted Fourier transform $\mathcal{F} = \mathrm{d}_0\,\Phi$.

***Proposition* 9.20** *For any $v \in H^1_r$, the operator $\mathrm{d}_v\,\Phi - \mathcal{F} : H^1_r \to h^{1/2}_r$ is compact.*

Proof. Let $v \in H^1_r$ be given. Instead of considering $\mathrm{d}_v\,\Phi - \mathcal{F}$ we compose it with $\mathcal{F}^{-1}$, $\mathcal{F}^{-1}\circ \mathrm{d}_v\,\Phi - I$ and then conjugate it with $P : H^1_r \to L^2_r$ to obtain

$$P\circ\mathcal{F}^{-1}\circ \mathrm{d}_v\,\Phi\circ P^{-1} - Id : L^2_r \to L^2_r.$$

It is to show that this operator is compact. Recall that

$$\mathrm{d}\Phi[\mathring{v}] = \left((\langle 2n\rangle^{1/2}\langle \mathring{v}, b_n^+\rangle)_n,\ (\langle 2n\rangle^{1/2}\langle \mathring{v}, b_n^-\rangle)_n\right)$$

and

$$\mathcal{F}^{-1}\big((\mathring{x}_n)_n,\ (\mathring{y}_n)_n\big) = \sum_n \frac{\mathring{x}_n}{\langle 2n\rangle^{1/2}} d_n^+ + \frac{\mathring{y}_n}{\langle 2n\rangle^{1/2}} d_n^-.$$

Hence

$$\mathcal{F}^{-1} \circ \mathrm{d}\Phi[\mathring{v}] = \sum_{n\in\mathbb{Z}} \langle \mathring{v}, b_n^+\rangle d_n^+ + \langle \mathring{v}, b_n^-\rangle d_n^-.$$

For any given $\mathring{v} = (\mathring{q}, \mathring{p}) \in H_r^1$, let

$$\tilde{v} := P\mathring{v}, \quad \tilde{q} := P\mathring{q}, \quad \tilde{p} := P\mathring{p}.$$

Since P^{-1} is selfadjoint with respect to $\langle\cdot,\cdot\rangle_r$ one has $\langle P^{-1}\tilde{v}, b_n^{\pm}\rangle = \langle \tilde{v}, P^{-1}b_n^{\pm}\rangle$ and hence

$$P \circ \mathcal{F}^{-1} \circ \mathrm{d}_v\,\Phi \circ P^{-1}[\tilde{v}] = \sum_{n\in\mathbb{Z}} \langle \tilde{v}, P^{-1}b_n^+\rangle P d_n^+ + \langle \tilde{v}, P^{-1}b_n^-\rangle P d_n^-.$$

Note that $P\mathcal{F}^{-1}\circ \mathrm{d}_v\,\Phi\circ P^{-1}$ is a bounded linear operator on L_r^2 depending continuously on v. Its adjoint, denoted by A_v, is given by

$$A_v[\tilde{v}] = \sum_{n\in\mathbb{Z}} \langle \tilde{v}, P d_n^+\rangle P^{-1} b_n^+ + \langle \tilde{v}, P d_n^-\rangle P^{-1} b_n^-. \tag{9.25}$$

It suffices to show that $A_v - Id$ is compact. Note that $Pd_n^{\pm} = \langle 2n\rangle d_n^{\pm}$ and

$$\tilde{v} = \sum_n \langle \tilde{v}, d_n^+\rangle d_n^+ + \langle \tilde{v}, d_n^-\rangle d_n^-.$$

Hence

$$(A_v - Id)[\tilde{v}] = \sum_{n\in\mathbb{Z}} \langle \tilde{v}, d_n^+\rangle \langle 2n\rangle P^{-1}(b_n^+ - d_n^+) + \langle \tilde{v}, d_n^-\rangle \langle 2n\rangle P^{-1}(b_n^- - d_n^-).$$

To see that $A_v - Id$ is compact we write it as a sum of two operators, $A_v - Id = K_v^+ + K_v^-$ where

$$K_v^+[\tilde{v}] := \sum_{n\geq 0} \langle \tilde{v}, d_n^+\rangle \langle 2n\rangle P^{-1}(b_n^+ - d_n^+) + \langle \tilde{v}, d_n^-\rangle \langle 2n\rangle P^{-1}(b_n^- - d_n^-)$$

and

$$K_v^-[\tilde{v}] := \sum_{n< 0} \langle \tilde{v}, d_n^+\rangle \langle 2n\rangle P^{-1}(b_n^+ - d_n^+) + \langle \tilde{v}, d_n^-\rangle \langle 2n\rangle P^{-1}(b_n^- - d_n^-).$$

In case where v is a right sided finite gap potential in H_r^1, Lemma 9.19 implies that K_v^+ is a Hilbert-Schmidt operator,

$$\sum_{n\geq 0} \|\langle 2n\rangle P^{-1}(b_n^+ - d_n^-)\|^2 + \|\langle 2n\rangle P^{-1}(b_n^- - d_n^-)\|^2 < \infty.$$

Since $A_v - Id$ and hence K_v^+ depend continuously on v and by Theorem 4.15 the right sided finite gap potentials in H_r^1 are dense in H_r^1, it then follows that $K_v^+ : L_r^2 \to L_r^2$ is compact for any $v \in H_r^1$. Similarly, Lemma 9.19 implies that for any left sided finite gap potential v, the operator K_v^- is Hilbert-Schmidt and hence again, Theorem 4.15 yields that $K_v^- : L_r^2 \to L_r^2$ is compact for any $v \in H_r^1$. Altogether we have shown that $A_v - Id = K_v^+ + K_v^- : L_r^2 \to L_r^2$ is compact for any $v \in H_r^1$. By the discussion above, this implies the claimed statement. □

Next we want to investigate the injectivity of the operator $\mathrm{d}_v\,\Phi : H_r^1 \to h_r^{1/2}$ for $v \in H_r^1$, or equivalently of the bounded linear operator $A_v : L_r^2 \to L_r^2$, introduced in (9.25). Recall that for any $\tilde{v} \in L_r^2$,

$$A_v[\tilde{v}] = \sum_{n\in\mathbb{Z}} \langle \tilde{v}, d_n^+\rangle \langle 2n\rangle P^{-1} b_n^+ + \langle \tilde{v}, d_n^-\rangle \langle 2n\rangle P^{-1} b_n^-$$

where $b_n^+ = \langle 2n\rangle^{-1/2}\partial x_n$, $b_n^- = \langle 2n\rangle^{-1/2}\partial y_n$, and hence $P^{-1}b_n^{\pm} \in L_r^2$, and

$$d_n^+ = \big(-\cos(2\pi n x),\ \sin(2\pi n x)\big), \quad d_n^- = \big(-\sin(2\pi n x),\ -\cos(2\pi n x)\big).$$

Since A_v is a bounded operator,

$$\sum_{n\in\mathbb{Z}} \alpha_n^+ \langle 2n\rangle P^{-1} b_n^+ + \alpha_n^- \langle 2n\rangle P^{-1} b_n^-$$

converges in L^2_r for any sequences $(\alpha_n^+)_n$, $(\alpha_n^-)_n$ in $\ell^2_{\mathbb{R}}$. The operator A_v is one-to-one if any sequences $(\alpha_n^+)_n$, $(\alpha_n^-)_n$ in $\ell^2_{\mathbb{R}}$, with

$$\sum_{n\in\mathbb{Z}} \alpha_n^+ \langle 2n\rangle P^{-1} b_n^+ + \alpha_n^- \langle 2n\rangle P^{-1} b_n^- = 0$$

must both vanish. If this property holds we say that $\langle 2n\rangle P^{-1} b_n^+$, $\langle 2n\rangle P^{-1} b_n^-$, $n \in \mathbb{Z}$, are linearly independent in L^2_r. First we ned to establish some auxiliary result. For any $v \in H^1_r$, define

$$\mathcal{S} = \mathcal{S}_v := \{\, n \in \mathbb{Z} : \ \lambda_n^-(v) < \lambda_n^+(v) \,\}.$$

Lemma 9.21 *Let $v \in H^1_r$. If $\frac{\langle 2n\rangle^{1/2}}{\sqrt{2I_n}} P^{-1}\partial I_n$, $n \in \mathcal{S}_v$, are linearly independent in L^2_r, then so are the vectors $\langle 2n\rangle P^{-1} b_n^+$, $\langle 2n\rangle P^{-1} b_n^-$, $n \in \mathbb{Z}$, and hence $\mathrm{d}_v\Phi : H^1_r \to h^{1/2}_r$ is one-to-one.*

Proof. FIx $v \in H^1_r$. Suppose that $(\alpha_n^+)_n$, $(\alpha_n^-)_n$ are sequences in $\ell^2_{\mathbb{R}}$ so that $f = 0$ where

$$f = \sum_{n\in\mathbb{Z}} \alpha_n^+ \langle 2n\rangle P^{-1} b_n^+ + \alpha_n^- \langle 2n\rangle P^{-1} b_n^-.$$

Recall that for any $n \in \mathbb{Z}$, $b_n^+ = \langle 2n\rangle^{-1/2}\partial x_n$ and $b_n^- = \langle 2n\rangle^{-1/2}\partial y_n$ and that $\langle 2n\rangle P^{-1} b_n^\pm$, $\partial I_n \in L^2_r$. By Corollary 9.11 it then follows that for any $m \in \mathcal{S}$

$$\begin{aligned}
0 =& \langle f, J\partial I_m\rangle \\
=& \sum_{n\in\mathbb{Z}} \alpha_n^+ \langle 2n\rangle^{1/2} \langle \partial x_n,\ JP^{-1}\partial I_m\rangle + \alpha_n^- \langle 2n\rangle^{1/2} \langle \partial y_n,\ JP^{-1}\partial I_m\rangle \\
=& \alpha_m^+ \langle 2m\rangle^{1/2}\{x_m, I_m\} + \alpha_m^- \langle 2m\rangle^{1/2}\{y_m, I_m\} \\
=& \langle 2m\rangle^{1/2}\big(\alpha_m^+ y_m - \alpha_m^- x_m\big).
\end{aligned}$$

It follows that the 2-vector (α_m^+, α_m^-) is parallel to the 2-vector $(x_m, y_m) = \sqrt{2I_m}\big(\cos(\theta_m), \sin(\theta_m)\big)$. Hence $(\alpha_m^+, \alpha_m^-) = a_m\big(\cos(\theta_m),\ \sin(\theta_m)\big)$ with $a_m \in \mathbb{R}$ satisfying $a_m^2 = (\alpha_m^+)^2 + (\alpha_m^-)^2$ and

$$\begin{aligned}
\langle 2m\rangle^{1/2}(\alpha_m^+ b_m^+ + \alpha_m^- b_m^-) =& \alpha_m^+\partial x_m + \alpha_m^-\partial y_m \\
=& a_m\big(\cos(\theta_m)\partial x_m + \sin(\theta_m)\partial y_m\big) \\
=& \frac{a_m}{\sqrt{2I_m}}(x_m\partial x_m + y_m\partial y_m) \\
=& \frac{a_m}{\sqrt{2I_m}}\partial I_m.
\end{aligned}$$

As a consequence

$$f = \sum_{n\in\mathcal{S}} \frac{a_n}{\sqrt{2I_n}}\langle 2n\rangle^{1/2} P^{-1}\partial I_n + \sum_{n\in\mathbb{Z}\setminus\mathcal{S}} \alpha_n^+ \langle 2n\rangle P^{-1} b_n^+ + \alpha_n^- \langle 2n\rangle P^{-1} b_n^-.$$

Note that by Corollary 9.11 for any $m \in \mathbb{Z}\setminus\mathcal{S}$ and $n \in \mathcal{S}$

$$\langle \partial x_m,\ JP^{-1}\partial I_n\rangle = \{x_m,\ I_n\} = 0$$

and

$$\langle \partial y_m,\ JP^{-1}\partial I_n\rangle = \{y_m,\ I_n\} = 0.$$

Furthermore, by Proposition 9.14, $\partial x_m \in L^2_r$ for any $m \in \mathbb{Z}\setminus\mathcal{S}$ and

$$0 = \langle \partial x_m, Jf\rangle = \alpha_m^- \langle 2m\rangle^{1/2} \langle \partial x_m,\ JP^{-1}\partial y_m\rangle = \alpha_m^- \langle 2m\rangle^{1/2}\{x_m,\ y_m\}.$$

Since $\{x_m,\ y_m\} \neq 0$ by Proposition 9.14 one has $\alpha_m^- = 0$. Similarly one derives that $\alpha_m^+ = 0$. Hence $f = \sum_{n\in\mathcal{S}} a_n \frac{\langle 2n\rangle^{1/2}}{\sqrt{2I_n}} P^{-1}\partial I_n$. Since by assumption the vectors $\frac{\langle 2n\rangle^{1/2}}{\sqrt{2I_n}} P^{-1}\partial I_n$, $n \in \mathcal{S}$, are linearly independent in L^2_r, $a_n = 0$ and hence $\alpha_n^+ = 0$, $\alpha_n^- = 0$ for any $n \in \mathcal{S}$. By the discussion ahead of the lemma it then follows that $\mathrm{d}_v\Phi$ is one-to-one. □

***Corollary* 9.22** *For any finite gap potenatial $v \in H^1_r$, $\mathrm{d}_v \Phi$ is one-to-one.*

Proof. By Lemma 9.21 it is to show that the vectors $\frac{\langle 2n\rangle^{1/2}}{\sqrt{2I_n}} P^{-1}\partial I_n$, $n \in \mathcal{S}_v$, are linearly independent in L^2_r. Since v is assumed to be a finite gap potential, $\mathcal{S}_v$ is finite. By Lemma 7.2, ∂I_n is in L^2_r implying that for any $n \in \mathcal{S}$, $\frac{\langle 2n\rangle^{1/2}}{\sqrt{2I_n}} P^{-1}\partial I_n$ is in H^1_r. Assume that $f := \sum_{n\in\mathcal{S}_v} a_n \frac{\langle 2n\rangle^{1/2}}{\sqrt{2I_n}} P^{-1}\partial I_n = 0$ for some real numbers $a_n \in \mathbb{R}$, $n \in \mathcal{S}_v$. Since $\partial\theta_m$, $m \in \mathcal{S}_v$, is in H^{-1}_r, we conclude that for any $m \in \mathcal{S}_v$, $\langle \partial\theta_m,\ \sum_{n\in\mathcal{S}_v} a_n \frac{\langle 2n\rangle^{1/2}}{\sqrt{2I_n}} JP^{-1}\partial I_n\rangle$ is well defined and must vanish. By Proposition 9.10, $\langle\partial\theta_m,\ JP^{-1}\partial I_n\rangle = \{\theta_m,\ I_n\} = -\delta_{mn}$ and hence

$$0 = \sum_{n\in\mathcal{S}_v} a_n \frac{\langle 2n\rangle^{1/2}}{\sqrt{2I_n}}\{\theta_m,\ I_n\} = -a_m \frac{\langle 2m\rangle^{1/2}}{\sqrt{2I_m}}.$$

It follows that $a_m = 0$ for any $m \in \mathcal{S}_v$, proving that $\frac{\langle 2n\rangle^{1/2}}{\sqrt{2I_n}} P^{-1}\partial I_n$, $n \in \mathcal{S}_v$, are linearly independent. □

We now want to improve on Corollary 9.22 by showing that $\mathrm{d}_v \Phi$ is one-to-one for a larger class of potentials in H^1_r. Recall that for any $v \in H^1_r$, the Dirichlet eigenvalues μ_k, $k \in \mathbb{Z}$, satisfy $\lambda^-_k \le \mu_k \le \lambda^+_k$ for any $k \in \mathbb{Z}$, i.e.

$$|\mu_k - \tau_k| \le \gamma_k/2 \quad \forall\ k \in \mathbb{Z}.$$

By Proposition 7.14, for any sequence $(\nu_n)_n$ with $\lambda^-_n \le \nu_n \le \lambda^+_n$, $n \in \mathbb{Z}$, there exists a potential in $\mathrm{Iso}(v)$ so that the corresponding Dirichlet eigenvalues are $(\nu_n)_n$.

***Lemma* 9.23** *Assume that for a given $v \in H^1_r$ there exists $C > 0$ so that*

$$|\mu_k - \tau_k| \le C\gamma_k^2 \quad \forall\ k \in \mathbb{Z}.$$

Then $\mathrm{d}_v \Phi : H^1_r \to h^{1/2}_r$ is one-to-one.

Proof. Fir $v \in H^1_r$ and let $\mathcal{S} \equiv \mathcal{S}_v$. By Lemma 9.21 it is to show that $\frac{\langle 2k\rangle^{1/2}}{\sqrt{2I_k}} JP^{-1}\partial I_k$, $k \in \mathcal{S}$, are linearly independent. Since $\sqrt{I_k} = \xi_k \frac{\gamma_k}{\sqrt{\tau_k}} = O(\frac{\gamma_k}{\langle k\rangle^{1/2}})$ it suffices to show that $\frac{\langle k\rangle}{\gamma_k} JP^{-1}\partial I_k$, $k \in \mathcal{S}$, are linearly independent. Assume that for some sequence $(\alpha_n)_{n\in\mathcal{S}} \in \ell^2(\mathcal{S},\mathbb{R})$, $f := \sum_{k\in\mathcal{S}} \alpha_k \frac{\langle k\rangle}{\gamma_k} JP^{-1}\partial I_k = 0$. Where we recall that the series converges in L^2_r. It is to show that $\alpha_k = 0$ for any $k \in \mathcal{S}$. Since $\partial I_k \in L^2_r$ (cf Lemma 7.2), $JP^{-1}\partial I_k \in H^1_r$. In case the series $\sum_{k\in\mathcal{S}} \alpha_k \frac{\langle k\rangle}{\gamma_k} JP^{-1}\partial I_k$ converges in H^1_r, one can take the inner product with $\partial\theta_n \in H^{-1}_r$ for any $n \in \mathcal{S}$ to conclude from the canonical relations $\{\theta_n,\ I_k\} = -\delta_{nk}$ that

$$0 = \langle\partial\theta_n,\ f\rangle = \sum_{k\in\mathcal{S}} \alpha_k \frac{\langle k\rangle}{\gamma_k}\langle\partial\theta_n,\ JP^{-1}\partial I_k\rangle = \sum_{k\in\mathcal{S}} \alpha_k \frac{\langle k\rangle}{\gamma_k}\{\theta_n,\ I_k\} = -\alpha_n \frac{\langle n\rangle}{\gamma_n}$$

implying that $\alpha_n = 0$. In the case the series $\sum_{k\in\mathcal{S}} \alpha_k \frac{\langle k\rangle}{\gamma_k} JP^{-1}\partial I_k$ converges only in L^2_r where elaborate arguments are needed. Let us consider $n \ge 0$ and recall that

$$\theta_n = \sum_{m\in\mathbb{Z}} \beta^n_{1,m} + \sum_{m\in\mathbb{Z}} \beta^n_{2,m}, \quad \beta^n_{1,n} := \eta_n.$$

Introduce

$$\mathcal{S}_1 := \{\ m \in \mathbb{Z} :\ |m| \in \mathcal{S}\ \}$$

and consider $\beta^n_{1,m}$ with $m \in \mathcal{S}_1$. Assume for the moment that

$$\lambda^-_{1,m} < \mu_{1,m} < \lambda^+_{1,m}$$

and that $f = \sum_{k\in\mathcal{S}} \alpha_k \frac{\langle k\rangle}{\gamma_k} JP^{-1}\partial I_k$ converges in H^1_r. Then

$$\langle\partial\beta^n_{1,m},\ f\rangle = \sum_{k\in\mathcal{S}} \alpha_k \frac{\langle k\rangle}{\gamma_k}\{\beta^n_{1,m},\ I_k\}.$$

Since

$$\partial I_k = -\frac{4}{\pi}\int_{\Gamma_k} \frac{1}{\lambda} \frac{\partial\Delta(\lambda)}{\sqrt[c]{\chi_p(\lambda)}}\,\mathrm{d}\lambda$$

one gets

$$\langle \beta^n_{1,m},\ f\rangle = \sum_{k\in\mathcal{S}} \alpha_k \frac{\langle k\rangle}{\gamma_k}\frac{(-4)}{\pi}\int_{\Gamma_k}\frac{1}{\lambda}\{\beta^n_{1,m},\ \Delta_\lambda\}\frac{\mathrm{d}\lambda}{\sqrt[c]{\chi_p(\lambda)}}.$$

Recall from (9.18) that since by assumption $\lambda^-_{1,m} < \mu_{1,m} < \lambda^+_{1,m}$

$$\{\beta^n_{1,m},\ \Delta_\lambda\} = \frac{\psi_n(\mu_{1,m})}{\sqrt[*]{\chi_p(\mu_{1,m})}}\{\mu_{1,m},\ \Delta_\lambda\}$$

implying that

$$\begin{aligned}\langle \beta^n_{1,m},\ f\rangle =& \sum_{k\in\mathcal{S}} \alpha_k \frac{\langle k\rangle}{\gamma_k}\frac{(-4)}{\pi}\int_{\Gamma_k}\frac{1}{\lambda}\frac{\psi_n(\mu_{1,m})}{\sqrt[*]{\chi_p(\mu_{1,m})}}\{\mu_{1,m},\ \Delta_\lambda\}\frac{\mathrm{d}\lambda}{\sqrt[c]{\chi_p(\lambda)}}\\ =& \sum_{k\in\mathcal{S}} \alpha_k \frac{\langle k\rangle}{\gamma_k}\frac{\psi_n(\mu_{1,m})}{\sqrt[*]{\chi_p(\mu_{1,m})}}\{\mu_{1,m},\ I_k\}.\end{aligned}$$

Since by Lemma 7.7, $\partial\mu_{1,m} \in L^2_r$ the inner product of $\frac{\psi_n(\mu_{1,m})}{\sqrt[*]{\chi_p(\mu_{1,m})}}\partial\mu_{1,m}$ with f is well defined even if the series defining f converges only in L^2_r. Hence we have

$$0 = \sum_{k\in\mathcal{S}} \alpha_k \frac{\langle k\rangle}{\gamma_k}\frac{\psi_n(\mu_{1,m})}{\sqrt[*]{\chi_p(\mu_{1,m})}}\int_{\Gamma_k}\frac{8}{\lambda}\{\mu_{1,m},\ \Delta_\lambda\}\frac{\mathrm{d}\lambda}{\sqrt[c]{\chi_p(\lambda)}}.$$

Since $\chi_D(\lambda)/(\lambda \pm \mu_{1,m})$ extend continuously to $\lambda = \pm\mu_{1,m}$ it follows from (9.15) that

$$\frac{8}{\lambda}\{\mu_{1,m},\ \Delta_\lambda\} = \frac{\delta(\mu_{1,m})}{\dot\chi_D(\mu_{1,m})}\Big(\frac{\chi_D(\lambda)}{\lambda-\mu_{1,m}} - \frac{\chi_D(\lambda)}{\lambda+\mu_{1,m}}\Big)$$

yielding, together with $\delta(\mu_{1,m}) = \sqrt[*]{\chi_p(\mu_{1,m})}$, $\sum_{k\in\mathcal{S}} q_{1,mk} = 0$ where

$$q_{1,mk} := \alpha_k \frac{\langle k\rangle}{\gamma_k}\frac{\psi_n(\mu_{1,m})}{\dot\chi_D(\mu_{1,m})}\frac{1}{2\pi}\int_{\Gamma_k}\big(\frac{\chi_D(\lambda)}{\lambda-\mu_{1,m}} - \frac{\chi_D(\lambda)}{\lambda+\mu_{1,m}}\big)\frac{\mathrm{d}\lambda}{\sqrt[c]{\chi_p(\lambda)}}.$$

The expression $q_{1,mk}$ is well defined even if $\mu_{1,m} \in \{\lambda^\pm_{1,m}\}$, hence by a limiting argument (cf Lemma 7.13, Corollary 7.15), we conclude that for any $v \in H^1_r$, one has $\sum_{k\in\mathcal{S}} q_{1,mk} = 0$. Furthermore, since $\psi_n(\mu_{1,m}) = 0$ for any $m \in \mathbb{Z}\setminus\mathcal{S}_1$, $q_{1,mk}$ is well defined for any $m\in\mathbb{Z}$ and $k\in\mathcal{S}$ and one has

$$\sum_{k\in\mathcal{S}} q_{1,mk} = 0 \quad \forall\ m \in \mathbb{Z}.$$

Similarly, we argue for $\beta^n_{2,m}$ for $m \in \mathcal{S}_2$ where $\mathcal{S}_2 := \{\ m\in\mathcal{S}\ :\ \ -|m| \in \mathcal{S}\ \}$. Assume for the moment that

$$\lambda^-_{2,m} < \mu_{2,m} < \lambda^+_{2,m}$$

and that $f = \sum_{k\in\mathcal{S}} \alpha_k \frac{\langle k\rangle}{\gamma_k} JP^{-1}\partial I_k$ converges in H^1_r. Then

$$\langle \beta^n_{2,m},\ f\rangle = \sum_{k\in\mathcal{S}} \alpha_k \frac{\langle k\rangle}{\gamma_k}\{\beta^n_{2,m},\ I_k\} = \sum_{k\in\mathcal{S}} \alpha_k \frac{\langle k\rangle}{\gamma_k}\frac{(-4)}{\pi}\int_{\Gamma_k}\frac{1}{\lambda}\{\beta^n_{2,m},\ \Delta_\lambda\}\frac{\mathrm{d}\lambda}{\sqrt[c]{\chi_p(\lambda)}}.$$

Since

$$\beta^n_{2,m} = \int_{(-16\lambda^-_{2,m})^{-1}}^{\kappa_{2,m}} \frac{\psi_n(\lambda)}{\sqrt[*]{\chi_p(\lambda)}}\,\mathrm{d}\lambda, \quad \kappa_{2,m} = (-16\mu_{2,m})^{-1}$$

one gets in the case $\lambda^-_{2,m} < \mu_{2,m} < \lambda^+_{2,m}$

$$\{\beta^n_{2,m},\ \Delta_\lambda\} = \frac{\psi_n(\kappa_{2,m})}{\sqrt[*]{\chi_p(\kappa_{2,m})}}\{\kappa_{2,m},\ \Delta_\lambda\}$$

implying that

$$\langle \beta^n_{2,m},\ f\rangle = \sum_{k\in\mathcal{S}} \alpha_k \frac{\langle k\rangle}{\gamma_k}\frac{(-4)}{\pi}\int_{\Gamma_k}\frac{1}{\lambda}\frac{\psi_n(\kappa_{2,m})}{\sqrt[*]{\chi_p(\kappa_{2,m})}}\{\kappa_{2,m},\ \Delta_\lambda\}\frac{\mathrm{d}\lambda}{\sqrt[c]{\chi_p(\lambda)}} = \sum_{k\in\mathcal{S}} \alpha_k \frac{\langle k\rangle}{\gamma_k}\frac{\psi_n(\kappa_{2,m})}{\sqrt[*]{\chi_p(\kappa_{2,m})}}\{\kappa_{2,m},\ I_k\}.$$

Since by Lemma 7.7, $\partial\kappa_{2,m} \in L^2_r$ the inner product of $\frac{\psi_n(\kappa_{2,m})}{\sqrt[*]{\chi_p(\kappa_{2,m})}}\partial\kappa_{2,m}$ with f is well defined even if the series defining f converges only in L^2_r. Hence we have

$$0 = \sum_{k\in\mathcal{S}} \alpha_k \frac{\langle k\rangle}{\gamma_k} \frac{\psi_n(\kappa_{2,m})}{\sqrt[*]{\chi_p(\kappa_{2,m})}} \int_{\Gamma_k} \frac{8}{\lambda}\{\kappa_{2,m},\ \Delta_\lambda\} \frac{\mathrm{d}\lambda}{\sqrt[c]{\chi_p(\lambda)}}.$$

By (9.16)

$$\frac{8}{\lambda}\{\kappa_{2,m},\ \Delta_\lambda\} = \frac{\delta(\kappa_{2,m})}{\dot\chi_D(\kappa_{2,m})}\Big(\frac{\chi_D(\lambda)}{\lambda-\kappa_{2,m}} - \frac{\chi_D(\lambda)}{\lambda+\kappa_{2,m}}\Big)$$

yielding together with $\delta(\kappa_{2,m}) = \sqrt[*]{\chi_p(\kappa_{2,m})}$

$$\sum_{k\in\mathcal{S}} q_{2,mk} = 0$$

where

$$q_{2,mk} := \alpha_k \frac{\langle k\rangle}{\gamma_k} \frac{\psi_n(\kappa_{2,m})}{\dot\chi_D(\kappa_{2,m})} \frac{1}{2\pi} \int_{\Gamma_k} \Big(\frac{\chi_D(\lambda)}{\lambda-\kappa_{2,m}} - \frac{\chi_D(\lambda)}{\lambda+\kappa_{2,m}}\Big)\frac{\mathrm{d}\lambda}{\sqrt[c]{\chi_p(\lambda)}}.$$

Again, $q_{2,ml}$ is well defined even if $\mu_{2,m} \in \{\lambda^\pm_{2,m}\}$ (or equivalently, $\kappa_{2,m} \in \{(-16\lambda^\pm_{2,m})^{-1}\}$). Hence by a limiting argument (cf Lemma 7.13, Corollary 7.15) we conclude that for any $v \in H^1_r$, one has $\sum_{k\in\mathcal{S}} q_{2,mk} = 0$. Furthermore, since $\psi_n(\kappa_{2,m}) = 0$ for any $m \in \mathbb{Z}\setminus\mathcal{S}_2$, $q_{2,mk}$ is well defined for any $m\in\mathbb{Z}$ and $k\in\mathcal{S}$ and one has

$$\sum_{k\in\mathcal{S}} q_{2,mk} = 0 \quad \forall\, m\in\mathbb{Z}.$$

By Lemma 9.24 below,

$$\sum_{m\in\mathbb{Z},k\in\mathcal{S}} |q_{j,mk}| < \infty, \quad j = 1,2.$$

Hence the summation over k and m in $\sum q_{j,mk}$ can be interchanged and we get similarly as in the proof of Lemma 9.9,

$$0 = \sum_{k\in\mathcal{S}} \alpha_k \frac{\langle k\rangle}{\gamma_k} \frac{1}{2\pi} \int_{\Gamma_k} (I_{n,\lambda} - II_{n,\lambda}) \frac{\mathrm{d}\lambda}{\sqrt[c]{\chi_p(\lambda)}}$$

where

$$I_{n,\lambda} := \sum_{m\in\mathbb{Z}} \frac{\psi_n(\mu_{1,m})}{\dot\chi_D(\mu_{1,m})}\frac{\chi_D(\lambda)}{\lambda-\mu_{1,m}} + \frac{\psi_n(\kappa_{2,m})}{\dot\chi_D(\kappa_{2,m})}\frac{\chi_D(\lambda)}{\lambda-\kappa_{2,m}}$$

and

$$II_{n,\lambda} := \sum_{m\in\mathbb{Z}} \frac{\psi_n(\mu_{1,m})}{\dot\chi_D(\mu_{1,m})}\frac{\chi_D(\lambda)}{\lambda+\mu_{1,m}} + \frac{\psi_n(\kappa_{2,m})}{\dot\chi_D(\kappa_{2,m})}\frac{\chi_D(\lambda)}{\lambda+\kappa_{2,m}}.$$

According to the proof of Lemma 9.9, $I_{n,\lambda} = \psi_n(\lambda)$ and $II_{n,\lambda} = \psi_n(-\lambda)$. We then apply Theorem 8.12 (cf also proof of Proposition 9.10) to conclude that

$$0 = \sum_{k\in\mathcal{S}} \alpha_k \frac{\langle k\rangle}{\gamma_k} \frac{1}{2\pi} \int_{\Gamma_k} (\psi_n(\lambda) - \psi_n(-\lambda)) \frac{\mathrm{d}\lambda}{\sqrt[c]{\chi_p(\lambda)}} = \alpha_n \frac{\langle n\rangle}{\gamma_n}.$$

We thus have proved that for any $n\in\mathcal{S}$ with $n\ge 0$, $\alpha_n = 0$. It remains to prove that $\alpha_{-n} = 0$ for any $-n\in\mathcal{S}$ with $n\ge 1$. Fortunately this follows easily from the symmetry properties of the action variables. By Proposition 8.6(ii),

$$I_{-k}(q,p) = I_k(-q,p) \quad k\in\mathcal{S}$$

and hence

$$(\partial_q I_{-k},\ \partial_p I_k)\big|_{q,p} = (-\partial_q I_k, \partial_p I_k)\big|_{-q,p}.$$

Going trough the arguments of the above proof for the potential $(-q,p)$ one then concludes that $\alpha_{-n} = 0$ for any $-n\in\mathcal{S}$ with $n\ge 1$. □

To complete the proof of Lemma 9.23 we need to show the following

Lemma 9.24 *Assume that for a given $v\in H^1_r$, there exists $C>0$ so that $|\mu_l - \tau_k| \le C\gamma_k^2 \ \ \forall\, k\in\mathbb{Z}$, then $\sum_{\substack{k\in\mathcal{S}\\ m\in\mathbb{Z}}} |q_{j,mk}| < \infty$ for $j=1,2$. Here $(q_{j,mk})_{m,k}$ are the sequences defined in the proof of Lemma 9.23.*

Remark 9.25. The asymptotic estimate $|\mu_k - \tau_k| = O(\gamma_k^2)$ is used to show that $\sum_{k\in\mathcal{S}} |q_{1,kk}| < \infty$.

Proof. Let us first estimate $|q_{1,mk}|$. Since $q_{1,mk} = 0$ for any $m \in \mathbb{Z}\setminus\mathcal{S}_1$ we will only consider $m \in \mathcal{S}_1$. (Recall that $\mathcal{S}_1 = \{\, l \in \mathbb{Z} : \ |l| \in \mathcal{S} \,\}$.) Recall that

$$q_{1,mk} = \alpha_k \frac{\langle k\rangle}{\gamma_k}\frac{\psi_n(\mu_{1,m})}{\dot\chi_D(\mu_{1,m})}\frac{1}{2\pi}\int_{\Gamma_k}\Big(\frac{\chi_D(\lambda)}{\lambda-\mu_{1,m}} - \frac{\chi_D(\lambda)}{\lambda+\mu_{1,m}}\Big)\frac{\mathrm{d}\lambda}{\sqrt[c]{\chi_p(\lambda)}}.$$

By standard asymptotic estimates, $|\psi_n(\mu_{1,m})| = O(\frac{|\sigma^n_{1,m}-\mu_{1,m}|}{\langle m-n\rangle})$ for $m \neq n$ and $|\psi_n(\mu_{1,n})| = O(1)$. Since by Theorem 8.12, $|\sigma^n_{1,m} - \tau_{1,m}| = O(\gamma_{1,m}^2)$ and by assumption $|\mu_{1,m} - \tau_{1,m}| = O(\gamma_{1,m}^2)$ one has

$$\psi_n(\mu_{1,m}) = \begin{cases} O(\frac{\gamma_{1,m}^2}{\langle m-n\rangle}) & \text{if } m \neq n \\ O(1) & \text{if } m = n \end{cases}.$$

Furthermore, again by standard asymptotic estimates $\dot\chi_D(\mu_{1,m}) = O(1)$. To estimate

$$\int_{\Gamma_k}\Big(\frac{\chi_D(\lambda)}{\lambda-\mu_{1,m}} - \frac{\chi_D(\lambda)}{\lambda+\mu_{1,m}}\Big)\frac{\mathrm{d}\lambda}{\sqrt[c]{\chi_p(\lambda)}} \qquad k \in \mathcal{S}_+ = \{\, k\in\mathcal{S} : \ k \geq 0 \,\}$$

we first rewrite $\frac{1}{\lambda-\mu_{1,m}} - \frac{1}{\lambda+\mu_{1,m}}$ in such a way that we get decay in $\lambda \sim k\pi$ in exchange for growth in $\mu_{1,m} \sim m\pi$

$$\begin{aligned}\frac{\lambda}{\mu_{1,m}}\Big(\frac{1}{\lambda-\mu_{1,m}} - \frac{1}{\lambda+\mu_{1,m}}\Big) &= \frac{\lambda}{\mu_{1,m}}\Big(\frac{2\mu_{1,m}}{(\lambda-\mu_{1,m})(\lambda+\mu_{1,m})}\Big)\\ &= \frac{2\lambda}{(\lambda-\mu_{1,m})(\lambda+\mu_{1,m})} = \frac{1}{\lambda-\mu_{1,m}} + \frac{1}{\lambda+\mu_{1,m}}.\end{aligned}$$

To estimate $\int_{\Gamma_k}\frac{\mu_{1,m}}{\lambda}\big(\frac{\chi_D(\lambda)}{\lambda-\mu_{1,m}} + \frac{\chi_D(\lambda)}{\lambda+\mu_{1,m}}\big)\frac{\mathrm{d}\lambda}{\sqrt[c]{\chi_p(\lambda)}}$ for $k \in \mathcal{S}_+$ deform the contour Γ_k to the interval $[\lambda_k^-, \lambda_k^+]$ and make the standard substitution $\lambda(t) = \tau_k + t\gamma_k/2$, $-1 \leq t \leq 1$ to get for $k \neq m$

$$\Big|\int_{\Gamma_k}\frac{\mu_{1,m}}{\lambda}\frac{\chi_D(\lambda)}{\lambda\mp\mu_{1,m}}\frac{\mathrm{d}\lambda}{\sqrt[c]{\chi_p(\lambda)}}\Big| = O\Big(\frac{\langle m\rangle}{\langle k\rangle}\frac{\gamma_k}{\langle k\pm m\rangle}\Big)$$

whereas in the case $k = m$ (hence $m \geq 0$)

$$\Big|\int_{\Gamma_m}\frac{\mu_{1,m}}{\lambda}\frac{\chi_D(\lambda)}{\lambda-\mu_{1,m}}\frac{\mathrm{d}\lambda}{\sqrt[c]{\chi_p(\lambda)}}\Big| = O(1).$$

We then get for any $m \in \mathcal{S}_1$ with $m \neq n$

$$\sum_{k\in\mathcal{S}_+}|q_{1,mk}| \leq C\sum_{\substack{k\in\mathcal{S}_+\\ k\neq m}}|\alpha_k|\frac{\langle k\rangle}{\gamma_k}\frac{\gamma_{1,m}^2}{\langle n-m\rangle}\frac{\langle m\rangle}{\langle k\rangle}\gamma_k\Big(\frac{1}{\langle k-m\rangle} + \frac{1}{\langle k+m\rangle}\Big) + C|\alpha_{|m|}|\frac{\langle m\rangle}{\gamma_{|m|}}\frac{\gamma_{|m|}^2}{\langle |m|-n\rangle}$$

or

$$\sum_{\substack{m\in\mathcal{S}_1\\ m\neq n}}\sum_{k\in\mathcal{S}_+}|q_{1,mk}| \leq C\sum_{\substack{m\in\mathcal{S}_1\\ m\neq n}}\sum_{\substack{k\in\mathcal{S}_+\\ k\neq m}}|\alpha_k|\gamma_{1,m}^2\Big(\frac{1}{\langle k-m\rangle} + \frac{1}{\langle k+m\rangle}\Big) + C\sum_{m\in\mathcal{S}_+}|\alpha_m|\gamma_m < \infty.$$

For $m = n$, one gets

$$\sum_{k\in\mathcal{S}_+}|q_{1,nk}| \leq C\sum_{\substack{k\in\mathcal{S}_+\\ k\neq n}}|\alpha_k|\frac{\langle k\rangle}{\gamma_k}\frac{\langle n\rangle}{\langle k\rangle}\gamma_k\Big(\frac{1}{\langle k-n\rangle} + \frac{1}{\langle k+n\rangle}\Big) + C|\alpha_n|\frac{\langle n\rangle}{\gamma_n}\gamma_n < \infty.$$

Next we estimate $\sum_{m\in\mathcal{S}_1}\sum_{k\in\mathcal{S}_-}|q_{1,mk}|$ where $\mathcal{S}_- = \{\, k\in\mathcal{S} : \ k \leq -1 \,\}$. Recall that for $k \leq -1$, $\Gamma_k = \Gamma_{2,k}$. Let $\tilde\Gamma_{2,k} := \{\, -\frac{1}{16\lambda} : \ \lambda \in \Gamma_{2,k} \,\}$. With the change of variable $\lambda = (-16\mu)^{-1}$ one obtains for $m \in \mathcal{S}_1$

$$\begin{aligned}&\int_{\Gamma_{2,k}}\chi_D(\lambda)\Big(\frac{1}{\lambda-\mu_{1,m}} - \frac{1}{\lambda+\mu_{1,m}}\Big)\frac{\mathrm{d}\lambda}{\sqrt[c]{\chi_p(\lambda)}}\\ &\qquad = \int_{\tilde\Gamma_{2,k}}\chi_D\Big(-\frac{1}{16\mu}\Big)\Big(\frac{1}{-\frac{1}{16\mu}-\mu_{1,m}} - \frac{1}{-\frac{1}{16\mu}+\mu_{1,m}}\Big)\frac{1}{\sqrt[c]{\chi_p(-\frac{1}{16\mu})}}\frac{1}{16\mu^2}\,\mathrm{d}\mu.\end{aligned}$$

By Lemma 6.13, $\chi_D(\lambda) = -\chi_{D,1}(\lambda)\chi_{D,2}(\lambda)/\chi_{D,2}(\infty)$ where $\chi_{D,1}(-\frac{1}{16\mu}) = O(1)$ and

$$\chi_{D,2}(-\frac{1}{16\mu}) = \prod_{l\in\mathbb{Z}} \frac{\mu_{2,l}-\mu}{\pi_l} = O(\gamma_k).$$

Clearly, $\frac{1}{-\frac{1}{16\mu}\pm\mu_{1,m}} = O(\frac{1}{\langle m\rangle})$, $\frac{1}{\mu^2} = O(\frac{1}{\langle k\rangle^2})$ and $\frac{1}{\sqrt[c]{\chi_p(-\frac{1}{16\mu})}} = O(\frac{1}{\sqrt{(\lambda^+_{2,k}-\mu)(\lambda^-_{2,k}-\mu)}})$. Hence

$$\Big|\int_{\Gamma_{2,k}} \chi_D(\lambda)\big(\frac{1}{\lambda-\mu_{1,m}} - \frac{1}{\lambda+\mu_{1,m}}\big)\frac{\mathrm{d}\lambda}{\sqrt[c]{\chi_p(\lambda)}}\Big| = O(\gamma_k\frac{1}{\langle m\rangle}\frac{1}{\langle k\rangle^2})$$

and altogether we get for $m\in\mathcal{S}_1$ with $m\neq n$

$$\sum_{k\in\mathcal{S}_-} |q_{1,mk}| \le C\sum_{\substack{k\in\mathcal{S}_-\\ k\neq m}} |\alpha_k|\frac{\langle k\rangle}{\gamma_k}\frac{\gamma^2_{1,m}}{\langle n-m\rangle}\gamma_k\frac{1}{\langle m\rangle}\frac{1}{\langle k\rangle^2} + C|\alpha_{-|m|}|\frac{\langle m\rangle}{\gamma_{-|m|}}\frac{\gamma^2_{-|m|}}{\langle -|m|-n\rangle}$$

whereas for $m=n$

$$\sum_{k\in\mathcal{S}_-} |q_{1,nk}| \le C\sum_{k\in\mathcal{S}_-} |\alpha_k|\frac{\langle k\rangle}{\gamma_k}\frac{1}{\langle n\rangle}\frac{\gamma_k}{\langle k\rangle^2} < \infty.$$

Hence

$$\sum_{\substack{k\in\mathcal{S}_-\\ m\in\mathcal{S}_1}} |q_{1,mk}| \le C\sum_{\substack{k\in\mathcal{S}_-\\ m\in\mathcal{S}_1\setminus\{n\}}} |\alpha_k|\frac{\gamma^2_{1,m}}{\langle n-m\rangle}\frac{1}{\langle m\rangle}\frac{1}{\langle k\rangle} + C\sum_{k\in\mathcal{S}_-}|\alpha_k|\frac{1}{\langle n\rangle}\frac{1}{\langle k\rangle} < \infty.$$

Altogether we thus have shown that

$$\sum_{\substack{k\in\mathcal{S}\\ m\in\mathbb{Z}}} |q_{1,mk}| = \sum_{\substack{k\in\mathcal{S}\\ m\in\mathcal{S}_1}} |q_{1,mk}| < \infty.$$

The proof of $\sum_{\substack{k\in\mathcal{S}\\ m\in\mathbb{Z}}} |q_{2,mk}| < \infty$ is similar. Since $q_{2,mk} = 0$ for any $m\in\mathbb{Z}\setminus\mathcal{S}_2$, we only consider $\sum_{\substack{k\in\mathcal{S}\\ m\in\mathcal{S}_2}} |q_{2,mk}|$ where $\mathcal{S}_2 = \{\, l\in\mathbb{Z} :\ -|l|\in\mathcal{S}\,\}$. Recall that for any $m\in\mathcal{S}_2$, $k\in\mathcal{S}$

$$q_{2,mk} = \alpha_k\frac{\langle k\rangle}{\gamma_k}\frac{\psi_n(\kappa_{2,m})}{\dot\chi_D(\kappa_{2,m})}\frac{1}{2\pi}\int_{\Gamma_k}\big(\frac{\chi_D(\lambda)}{\lambda-\kappa_{2,m}} - \frac{\chi_D(\lambda)}{\lambda+\kappa_{2,m}}\big)\frac{\mathrm{d}\lambda}{\sqrt[c]{\chi_p(\lambda)}}.$$

By standard asymptotic estimates,

$$|\psi_n(\kappa_{2,m})| = O(|\sigma^n_{2,m}-\mu_{2,m}|).$$

Since by Theorem 8.12, $|\sigma^n_{2,m}-\tau_{2,m}| = O(\gamma^2_{2,m})$ and by assumption $(\mu_{2,m}-\tau_{2,m}) = O(\gamma^2_{2,m})$ one has $\psi_n(\kappa_{2,m}) = O(\gamma^2_{2,m})$. Furthermore since $\chi_D(\lambda) = -\chi_{D,1}(\lambda)\chi_{D,2}(\lambda)/\chi_{D,2}(\infty)$ and $\kappa_{2,m}$ is a zero of $\chi_{D,2}(\lambda)$ we have $\dot\chi_D(\kappa_{2,m}) = -\chi_{D,1}(\kappa_{2,m})\dot\chi_{D,2}(\kappa_{2,m})/\chi_{D,2}(\infty)$. To compute $\dot\chi_{D,2}$ recall that $\chi_{D,2}$ has the product representation $\chi_{D,2}(\lambda) = \prod_{k\in\mathbb{Z}}\frac{\mu_{2,l}+\frac{1}{16\lambda}}{\pi_l}$. Since $\frac{1}{16\kappa_{2,m}} = -\mu_{2,m}$ and $\partial_x(\frac{1}{16\lambda})\big|_{\kappa_{2,m}} = -\frac{1}{16\kappa^2_{2,m}}$ one gets

$$\dot\chi_{D,2}(\kappa_{2,m}) = \big(\frac{1}{\pi_m}\prod_{l\neq m}\frac{\mu_{2,l}-\mu_{2,m}}{\pi_l}\big)\big(-\frac{1}{16\kappa^2_{2,m}}\big).$$

By standard asymptotic estimates one then concludes that

$$\Big|\frac{1}{\dot\chi_D(\kappa_{2,m})}\Big| = O(\kappa^2_{2,m}) = O(\frac{1}{\langle m\rangle^2}).$$

To estimate $\int_{\Gamma_k}\big(\frac{\chi_D(\lambda)}{\lambda-\kappa_{2,m}} - \frac{\chi_D(\lambda)}{\lambda+\kappa_{2,m}}\big)\frac{\mathrm{d}\lambda}{\sqrt[c]{\chi_p(\lambda)}}$ for $k\in\mathcal{S}_+$ note that by standard asymptotic estimates $\big|\frac{\chi_D(\lambda)}{\sqrt[c]{\chi_p(\lambda)}}\big| = O(\frac{\gamma_k}{w_{1,k}(\lambda)})$. Clearly one has $\big|\frac{1}{\lambda-\kappa_{2,m}} - \frac{1}{\lambda+\kappa_{2,m}}\big| = \big|\frac{2\kappa_{2,m}}{\lambda^2-\kappa^2_{2,m}}\big| = O(\frac{1}{\langle k\rangle^2}\frac{1}{\langle m\rangle})$ and hence

$$\Big|\int_{\Gamma_k}\big(\frac{\chi_D(\lambda)}{\lambda-\kappa_{2,m}} - \frac{\chi_D(\lambda)}{\lambda+\kappa_{2,m}}\big)\frac{\mathrm{d}\lambda}{\sqrt[c]{\chi_p(\lambda)}}\Big| = O(\frac{\gamma_k}{\langle k\rangle^2}\frac{1}{\langle m\rangle}).$$

We thus have proved that

$$\sum_{\substack{k\in\mathcal{S}_+\\ m\in\mathcal{S}_2}} |q_{2,mk}| \le C \sum_{\substack{k\in\mathcal{S}_+\\ m\in\mathcal{S}_2}} |\alpha_k| \frac{\langle k\rangle}{\gamma_k} \frac{\gamma_{2,m}^2}{\gamma_k} \frac{\gamma_k}{\langle k\rangle^2} \frac{1}{\langle m\rangle} < \infty.$$

To estimate $\int_{\Gamma_k} \big(\frac{\chi_D(\lambda)}{\lambda-\kappa_{2,m}} - \frac{\chi_D(\lambda)}{\lambda+\kappa_{2,m}} \big) \frac{d\lambda}{\sqrt[c]{\chi_p(\lambda)}}$ for $k \in \mathcal{S}_-$ we recall that $\Gamma_k = \Gamma_{2,k}$ for $k \le -1$. With $\tilde\Gamma_{2,k} := \{ -\frac{1}{16\lambda} : \lambda \in \Gamma_{2,k} \}$ and the change of variable $\lambda = (-16\mu)^{-1}$ one obtains for $m \in \mathcal{S}_2$

$$I_{k,m} := \int_{\Gamma_{2,k}} \big(\frac{\chi_D(\lambda)}{\lambda-\kappa_{2,m}} - \frac{\chi_D(\lambda)}{\lambda+\kappa_{2,m}} \big) \frac{d\lambda}{\sqrt[c]{\chi_p(\lambda)}} = \int_{\tilde\Gamma_{2,k}} \big(\frac{\chi_D(-\frac{1}{16\mu})}{-\frac{1}{16\mu}-\kappa_{2,m}} - \frac{\chi_D(-\frac{1}{16\mu})}{-\frac{1}{16\mu}+\kappa_{2,m}} \big) \frac{1}{\sqrt[c]{\chi_p(-\frac{1}{16\mu})}} \frac{d\mu}{16\mu^2}.$$

Furthermore, since

$$-\frac{1}{16\mu} \pm \kappa_{2,m} = \frac{\kappa_{2,m}}{\mu}(-\frac{1}{16\kappa_{2,m}} \pm \mu) = \frac{\kappa_{2,m}}{\mu}(\mu_{2,m} \pm \mu)$$

one gets

$$\kappa_{2,m} I_{k,m} = \int_{\tilde\Gamma_{2,k}} \big(\frac{\chi_D(-\frac{1}{16\mu})}{\mu_{2,m}-\mu} - \frac{\chi_D(-\frac{1}{16\mu})}{\mu_{2,m}+\mu} \big) \frac{1}{\sqrt[c]{\chi_p(-\frac{1}{16\mu})}} \frac{d\mu}{16\mu}$$

and hence by standard asymptotic estimates in case $k \ne m$

$$|\kappa_{2,m} I_{k,m}| \le C\big(\frac{\gamma_k}{\langle m-k\rangle} + \frac{\gamma_k}{\langle m+k\rangle} \big) \frac{1}{\langle k\rangle}$$

and for $k = m$, $|\kappa_{2,m} I_{m,m}| \le C \frac{1}{\langle m\rangle}$. Altogether we get

$$\sum_{\substack{k\in\mathcal{S}_-\\ m\in\mathcal{S}_2}} |q_{2,mk}| \le C \sum_{m\in\mathcal{S}_2} \sum_{\substack{k\in\mathcal{S}_-\\ k\ne m}} |\alpha_k| \frac{\langle k\rangle}{\gamma_k} \frac{\gamma_{2,m}^2}{\langle m\rangle} \big(\frac{1}{\langle m-k\rangle} + \frac{1}{\langle m+k\rangle} \big) \frac{\gamma_k}{\langle k\rangle} + C \sum_{\substack{m\in\mathcal{S}_2\\ m\le -1}} |\alpha_m| \frac{\langle m\rangle}{\gamma_m} \frac{\gamma_m^2}{\langle m\rangle} \frac{1}{\langle m\rangle} < \infty.$$

This completes the proof of Lemma 9.24. □

We summarize the resuts obtained in this section as follows :

***Theorem* 9.26** *For any $v \in H_r^1$ with the property that $\mu_k - \tau_k = O(\gamma_k^2)$, $d_v \Phi : H_r^1 \to h_r^{1/2}$ is a linear isomorphism. In particular, this is the case for any finite gap potential.*

Remark 9.27. By Proposition 7.14, for any $v \in H_r^1$, there exists $\tilde v \in \mathrm{Iso}(v)$ with the property that $\mu_k = \tau_k$ for any $k \in \mathbb{Z}$.

Proof. By Proposition 9.20, for any $v \in H_r^1$, $d_v \Phi$ is a Fredholm operator and by Lemma 9.23, for any $v \in H_r^1$ with $\mu_k - \tau_k = O(\gamma_k^2)$ as $k \to \infty$, $d_v \Phi$ is one-to-one. Hence for such potentials $d_v \Phi$ is a linear isomorphism. □

9.4 Birkhoff map near the origin

In this section we analyze the Birkhoff map near the origin and in the course of our investigation prove Theorem 1.1 as stated in the introduction. First we need to introduce some more notation. For any $(x^{(0)}, y^0) \in h_r^{1/2}$, let

$$\mathrm{Tor}(x^{(0)}, y^{(0)}) := \{ (x,y) \in h_r^{1/2} : x_n^2 + y_n^2 = (x_n^{(0)})^2 + (y_n^{(0)})^2 \ \ \forall\, n \in \mathbb{Z} \}.$$

Clearly $\mathrm{Tor}(x^{(0)}, y^{(0)})$ is a possibly infinite product of circles in $\mathbb{R}^2$ with radi $\sqrt{(x_n^{(0)})^2 + (y_n^{(0)})^2}$, centered at the origin.

By Proposition 9.17 $d_0\Phi : H_r^1 \to h_r^{1/2}$ is a linear isomorphism. By the inverse function theorem there exists $\rho' > 0$ so that $\Phi : B_{0,\rho'}^{H^1} \to \Phi(B_{0,\rho'}^{H^1})$ is a real analytic diffeormorphism. Choose $\rho > 0$ so that $\overline{B_{0,\rho}^{h^{1/2}}} \subseteq \Phi(B_{0,\rho'}^{H^1})$. Then

$$\Phi : V_{0,\rho} \to B_{0,\rho}^{h^{1/2}}, \quad V_{0,\rho} := \Phi^{-1}(B_{0,\rho}^{h^{1/2}}) \cap B_{0,\rho'}^{H^1}$$

is a real analytic diffeomorphism. Here for any Banach space E, $B^E_{0,\rho}$ denotes the open ball in E of radius ρ, centered at 0. First we investigate the isospectral set $\mathrm{Iso}(v)$ for a potential $v \in V_{0,\rho}$. Note that for any $(x,y) \in B^{h^{1/2}}_{0,\rho}$, $\mathrm{Tor}(x,y) \subseteq B^{h^{1/2}}_{0,\rho}$. By choosing ρ smaller if needed one can assure that for any $v_0 \in V_{0,\rho}$ and $v \in H^1_r$ with $H_{\sinh}(v_0) = H_{\sinh}(v)$ it follows that $v \in B^{H^1}_{0,\rho'}$ where we recall that $H_{\sinh}$ denotes the sinh-Gordon Hamiltonian (cf (2.50)).

The following result describes isospectral sets near 0 in terms of Birkhoff coordinates.

***Proposition* 9.28** *For any $v = (q,p) \in V_{0,\rho}$, $\mathrm{Iso}(v) \subseteq V_{0,\rho}$, $\Phi(\mathrm{Iso}(v)) = \mathrm{Tor}(\Phi(v))$, and $(-q,p) \in V_{0,\rho}$.*

Proof. Let $v_0 \in V_{0,\rho}$. Since for any $v \in \mathrm{Iso}(v_0)$, $\Delta(\lambda, v) = \Delta(\lambda, v_0)$ for any $\lambda \in \mathbb{C}^*$ and since the actions are defined entirely in terms of the Δ - function it follows that $I_n(v) = I_n(v_0)$ for any $n \in \mathbb{Z}$ and hence $\Phi(v) \in \mathrm{Tor}(\Phi(v_0))$. Thus

$$\Phi(\mathrm{Iso}(v)) \subseteq \mathrm{Tor}(\Phi(v)). \tag{9.26}$$

Since $H_{\sinh}$ is a spectral invariant (cf Theorem 2.20) it follows that $H_{\sinh}(v) = H_{\sinh}(v_0)$ and hence by the choice of ρ, $v \in B^{H^1}_{0,\rho'}$. Since $\Phi : B^{H^1}_{0,\rho'} \to \Phi(B^{H^1}_{0,\rho'})$ is one-to-one it then follows from (9.26) that $\mathrm{Iso}(v_0) \subseteq V_{0,\rho}$. To see that $\Phi(\mathrm{Iso}(v_0)) = \mathrm{Tor}(\Phi(v_0))$ we argue as follows. Let $\mathcal{S} = \{\, n \in \mathbb{Z} : \ I_n(v_0) \neq 0 \,\}$ and for any $n \in \mathbb{Z} \setminus \mathcal{S}$ consider the curve $t \mapsto (x(t), y(t))$ which moves counter clockwise around the n-th circle $x_n^2 + y_n^2 = 2I_n(v_0)$ of $\mathrm{Tor}(\Phi(v_0))$ with unit speed, while all other coordinates stay put. The preimage $v(t)$ of $(x(t), y(t))$ by Φ in $V_{0,\rho}$ is a smooth curve with nonvanishing velocity vector $v'(t) = \partial_t v(t)$. Note that $d_{v(t)}\Phi[v'(t)] = \big((d_{v(t)}x_k[v'(t)])_k,\ (d_{v(t)}y_k[v'(t)])_k\big)$ is given by

$$d_{v(t)}x_k[v'(t)] = 0, \quad d_{v(t)}y_k[v'(t)] = 0 \ \ \forall\ k \neq n$$

and

$$d_{v(t)}x_n[v'(t)] = x_n'(t) = -\frac{y_n(t)}{\sqrt{x_n(0)^2 + y_n(0)^2}}, \quad d_{v(t)}y_n[v'(t)] = y_n'(t) = \frac{x_n(t)}{\sqrt{x_n(0)^2 + y_n(0)^2}}.$$

We claim that $v'(t) = -JP^{-1}\partial_{v(t)}I_n$. Indeed, by Corollary 9.11,

$$d_{v(t)}x_k[-JP^{-1}\partial_{v(t)}I_n] = \{x_k,\, -I_n\}(v(t)) = 0 \ \ \forall\ k \neq n$$

$$d_{v(t)}y_k[-JP^{-1}\partial_{v(t)}I_n] = \{y_k,\, -I_n\}(v(t)) = 0 \ \ \forall\ k \neq n$$

$$d_{v(t)}x_n[-JP^{-1}\partial_{v(t)}I_n] = \{x_n,\, -I_n\}(v(t)) = -y_n(v(t))$$

$$d_{v(t)}y_n[-JP^{-1}\partial_{v(t)}I_n] = \{y_n,\, -I_n\}(v(t)) = x_n(v(t)).$$

Since $d_{v(t)}\Omega : H^1_r \to h^{1/2}_r$ is a linear isomorphism it follows that $v'(t) = -JP^{-1}\partial_{v(t)}I_n$. Since for any $\lambda \in \mathbb{C}^*$

$$\frac{\mathrm{d}}{\mathrm{d}t}\Delta_\lambda(v(t)) = \langle \Delta_\lambda(v(t)), -JP^{-1}\partial_{v(t)}I_n\rangle = -\{\Delta_\lambda,\, I_n\}(v(t))$$

it follows from Proposition 9.8 that Δ_λ is invariant along the curve $v(t)$. Hence $v(t) \in \mathrm{Iso}(v^\circ)$ for any $t \in \mathbb{R}$. Since this holds for any $n \in \mathbb{Z} \setminus \mathcal{S}$ we conclude by induction that

$$\Phi^{-1}(\mathrm{Tor}(\Phi(v^\circ))) \subseteq \mathrm{Iso}(v^\circ)).$$

Altogether we have proved that $\Phi(\mathrm{Iso}(v^\circ)) = \mathrm{Tor}(\Phi(v^\circ))$. It remains to prove that for any $v = (q,p) \in V_{0,\rho}$ one has $(-q,p) \in V_{0,\rho}$, i.e. that $V_{0,\rho}$ is invariant under the symmetry $\mathcal{S}_{rec}$. Recall that by Proposition 8.6(ii), $In(-q,p) = I_{-n}(q,p)$ for any $n \in \mathbb{Z}$, hence $\sum_{n\in\mathbb{Z}} I_n(-q,p) = \sum_{n\in\mathbb{Z}} I_n(q,p)$ implying that $((x_n(-q,p))_n, (y_n(-q,p))_n)$ and $((x_n(q,p))_n, (y_n(q,p))_n)$ have the same norm in $h^{1/2}_r$. Since $B^{h^{1/2}}_{0,\rho}$ is a ball it follows that $\Phi(-q,p) \in B^{h^{1/2}}_{0,\rho}$. Furthermore, one has $H_{\sinh}(-q,p) = H_{\sinh}(q,p)$ and hence $(-q,p) \in B^{H^1}_{0,\rho'}$. By our choice of ρ this implies that $(-q,p) \in V_{0,\rho}$. □

Proposition 9.28 allows to approximate $v \in V_{0,\rho}$ by finite gap potentials. To the best of our knowledge such a result has not been established previously. Recall that $v \in H^1_r$ is said to be a finite gap potential if $\mathcal{S} := \{\, n \in \mathbb{Z} : \ \gamma_n(v) = \lambda_n^+(v) - \lambda_n^-(v) > 0 \,\}$ is finite.

***Corollary* 9.29** *The set of finite gap potentials in $V_{0,\rho}$ is dense in $V_{0,\rho}$.*

Proof. Let $v \in V_{0,\rho}$ and $(x,y) = \Phi(v) \in B^{h^{1/2}}_{0,\rho}$. Define for any $N \geq 1$

$$x_n^N := \begin{cases} x_n & \forall\, |n| \leq N \\ 0 & \forall\, |n| > N, \end{cases} \qquad y_n^N := \begin{cases} y_n & \forall\, |n| \leq N \\ 0 & \forall\, |n| > N. \end{cases}$$

Then clearly $(x^N, y^N) \in B^{h^{1/2}}_{0,\rho}$ for any $N \geq 1$ and $v_N \in V_{0,\rho}$ with $\Phi(v_N) = (x^N, y^N)$ is a finite gap potential with $\mathcal{S} \subseteq \{\, n \in \mathbb{Z} \,:\, |n| \leq N \,\}$. Since $\lim_{N\to\infty}(x^N, y^N) = (x,y)$ in $B^{h^{1/2}}_{0,\rho}$, it follows that $\lim_{N\to\infty} v_N = v$. □

By construction, $V_{0,\rho}$ is the disjoint union of its isospectral sets, $V_{0,\rho} = \bigcup_{v \in V_{0,\rho}} \mathrm{Iso}(v)$. Using the characterization of $\mathrm{Iso}(v)$ in terms of Birkhoff coordinates it is straightforward to describe a section of this disjoint union which by a slight abuse of terminology is referred to as Lagrangian section of $V_{0,\rho}$. Define

$$V_{0,\rho}^{left} := \{\, v \in V_{0,\rho} \,:\, \mu_k = \lambda_k^- \;\; \forall\, k \in \mathbb{Z} \,\}, \quad V_{0,\rho}^{open} := \{\, v \in V_{0,\rho} \,:\, \lambda_k^- < \lambda_k^- \;\; \forall\, k \in \mathbb{Z} \,\}.$$

In view of Remark 8.32, Proposition 7.14 and Proposition 8.6(iii) the following holds.

***Corollary* 9.30** *(i)* $\Phi(V_{0,\rho}^{left}) = \{\, (x,y) \in \Phi(V_{0,\rho}) \,:\, y_k = 0,\, x_k \geq 0 \;\forall\, k \in \mathbb{Z} \,\}$ *Hence for any* $v \in V_{0,\rho}$, $\mathrm{Iso}(v) \cap V_{0,\rho}^{left}$ *consists of precisely one point.* $V_{0,\rho}$ *can be described by the coordinates* $x = (x_k)_{k\in\mathbb{Z}}$, $y = 0$ *where the* x_k*'s satisfy*

$$\Big(\sum_{k\in\mathbb{Z}} x_k^2\Big)^{1/2} < \rho, \quad x_k \geq 0 \;\; \forall\, k \in \mathbb{Z}.$$

(ii) $\Phi(V_{0,\rho}^{open}) = \{\, (x,y) \in \Phi(V_{0,\rho}) \,:\, x_k^2 + y_k^2 > 0 \;\; \forall\, k \in \mathbb{Z} \,\}$. *Hence* $V_{0,\rho}^{open}$ *is dense in* $V_{0,\rho}$ *and* $V_{0,\rho}^{open} \cap V_{0,\rho}^{left}$ *is dense in* $V_{0,\rho}^{left}$.

Finally we study the sinh-Gordon equation on $V_{0,\rho}$ in Birkhoff coordinates. First we need to make some preliminary considerations. Recall that for any $v = (q,p) \in H_r^1$,

$$H_{\sinh}(q,p) = \int_0^1 \Big(\frac{1}{2}(Pp)^2 + \frac{1}{2}(Pp)^2 + \cosh(q) - \frac{1}{2}q^2\Big)\,\mathrm{d}x.$$

By Corollary 7.6, $\{H_{\sinh},\, \Delta_\lambda\} = 0$ for any $\lambda \in \mathbb{C}^*$. Since $\partial I_n = -\frac{4}{\pi}\int_{\Gamma_n} \frac{1}{\lambda}\frac{\partial\Delta(\lambda)}{\sqrt[c]{\chi_p(\lambda)}}\,\mathrm{d}\lambda$ it then follows that $\{H_{\sinh},\, I_n\} = 0$ on H_r^1 for any $n \in \mathbb{Z}$ and hence by Proposition 9.28 that $\tilde{H} := H_{\sinh} \circ \Phi^{-1} : B^{h^{1/2}}_{0,\rho} \to \mathbb{R}$ only depends on the actions. Arguing as in [[8], Addendum to Theorem 15.1] it follows that for any $(x,y) = \big((x_n)_n,\, (y_n)_n\big) \in B^{h^{1/2}}_{0,\rho}$

$$\tilde{H}((x_n)_n,\, (y_n)_n) = H(((x_n^2 + y_n^2)/2)_n)$$

where H is a real analytic function on $\mathcal{C}^+_{0,\rho}$, defined by

$$\mathcal{C}^+_{0,\rho} := \{\, I = (I_n)_{n\in\mathbb{Z}} \,:\, I_n \geq 0 \;\; \forall\, n \in \mathbb{Z};\, \sum_{n\in\mathbb{Z}} \langle n \rangle I_n < \rho \,\}.$$

(Note that $\mathcal{C}^+_{0,\rho}$ is a subset of the positive quadrant in the weighted ℓ^1-sequence space $\ell^{1,1}(\mathbb{Z},\mathbb{R})$.) By the chain rule it then follows that

$$\partial H_{\sinh} = \sum_{n\in\mathbb{Z}} \omega_n \partial I_n, \quad \omega_n = \partial_{I_n} H \;\; \forall\, n \in \mathbb{Z}, \tag{9.27}$$

where the series converges in H_r^{-1}.

***Lemma* 9.31** *For any* $n \in \mathbb{Z}$, ∂I_n *vanishes on* $Z_n \cap H_r^1$.

Proof. By Proposition 8.6(i),

$$\partial I_n = -\frac{4}{\pi}\int_{\Gamma_n} \frac{1}{\lambda}\frac{\partial\Delta(\lambda)}{\sqrt[c]{\chi_p(\lambda)}}\,\mathrm{d}\lambda.$$

Since for any $v \in Z_n \cap H_r^1$, $\sqrt[c]{\chi_p(\lambda)}$ contains the factor $\lambda - \tau_n$, where $\tau_n = \lambda_n^- (= \lambda_n^+)$ one gets by Cauchy's theorem that ∂I_n is proportional to $\partial\Delta\big|_{\tau_n}$. Furthermore, on $Z_n \cap H_r^1$, $\grave{m}_2$, $\grave{m}_3$, and δ vanish at $\lambda = \tau_n$. By Lemma 5.2 one then concludes that $\partial\Delta\big|_{\tau_n} = 0$ and hence $\partial I_n = 0$ on $Z_n \cap H_r^1$. □

For any $\mathcal{S} \subseteq \mathbb{Z}$, denote by $\mathcal{G}_{\mathcal{S}} \subseteq H^1_r$ the set of all $\mathcal{S}$-gap potentials,

$$\mathcal{G}_{\mathcal{S}} := \{\ v \in H^1_r\ :\ \lambda^-_k(v) < \lambda^+_k(v) \text{ iff } k \in \mathcal{S}\ \}$$

and define $\mathcal{G}_{\mathcal{S},\rho} := \mathcal{G}_{\mathcal{S}} \cap V_{0,\rho}$. Correspondingly, introduce

$$g_{\mathcal{S}} := \{\ (x, y) \in h^{1/2}_r\ :\ x^2_k + y^2_k = 0 \text{ iff } k \in \mathcal{S}\ \}$$

and let $g_{\mathcal{S},\rho} := g_{\mathcal{S}} \cap B^{h^{1/2}}_{0,\rho}$. Then clearly,

$$\Phi(\mathcal{G}_{\mathcal{S},\rho}) = g_{\mathcal{S},\rho}.$$

Lemma 9.31 yields the following

***Corollary* 9.32** *For any $\mathcal{S} \subseteq \mathbb{Z}$ finite, $\partial H_{\sinh} = \sum_{n\in\mathcal{S}} \omega_n \partial I_n$ on $\mathcal{G}_{\mathcal{S},\rho}$ and the Hamiltonian vector field $JP^{-1}\partial H_{\sinh} = \sum_{n\in\mathcal{S}} \omega_n JP^{-1}\partial I_n$ is in H^1_r for any $v \in \mathcal{G}_{\mathcal{S},\rho}$. As a consequence, for any $n \in \mathbb{Z}$, the Poisson brackets $\{H_{\sinh},\, x_n\}$, $\{H_{\sinh},\, y_n\}$ are well defined on $\mathcal{G}_{\mathcal{S},\rho}$ and*

$$\{x_n,\, H_{\sinh}\} = \omega_n y_n, \quad \{y_n,\, H_{\sinh}\} = -\omega_n x_n.$$

Proof. For any $\mathcal{S} \subseteq \mathbb{Z}$ finite, one concludes from (9.27) and Lemma 9.31 that $\partial H_{\sinh} = \sum_{n\in\mathcal{S}} \omega_n \partial I_n$. Since by Lemma 7.2, $\partial I_n \in L^2_r$ the Hamiltonian vector field $JP^{-1}\partial H_{\sinh} = \sum_{n\in\mathcal{S}} \omega_n JP^{-1}\partial I_n$ is in H^1_r. As a consequence, for any $n \in \mathbb{Z}$, $\{x_n,\, H_{\sinh}\}$ and $\{y_n,\, H_{\sinh}\}$ are well defined on $\mathcal{G}_{\mathcal{S},\rho}$ and by Corollary 9.11

$$\{x_n,\, H_{\sinh}\} = \sum_{k\in\mathcal{S}} \omega_k \{x_n,\, I_k\} = \omega_n y_n$$

and

$$\{y_n,\, H_{\sinh}\} = \sum_{k\in\mathcal{S}} \omega_k \{y_n,\, I_k\} = -\omega_n x_n.$$

□

For any $(x^{(0)},\, y^{(0)}) \in B^{h^{1/2}}_{0,\rho}$ with

$$x^{(0)}_n = \sqrt{2I^{(0)}_n} \cos(\theta^{(0)}_n), \quad y^{(0)}_n = \sqrt{2I^{(0)}_n} \sin(\theta^{(0)}_n)$$

and any $t \in \mathbb{R}$, define

$$\mathcal{S}^{h^{1/2}}_t (x^{(0)},\, y^{(0)}) := (x(t),\, y(t))$$

where for any $n \in \mathbb{Z}$

$$x_n(t) := \sqrt{2I^{(0)}_n} \cos(-\omega_n t + \theta^{(0)}_n), \quad y_n(t) := \sqrt{2I^{(0)}_n} \sin(-\omega_n t + \theta^{(0)}_n)$$

and

$$\omega_n = \partial_{I_n} H\big|_{I^{(0)}}, \quad I^{(0)} = (I^{(0)}_n)_{n\in\mathbb{Z}}.$$

Note that

$$(x(t), y(t)) \in \operatorname{Tor}(x^{(0)}, y^{(0)}), \quad \forall\ t \in \mathbb{R}. \tag{9.28}$$

In addition the map

$$\mathbb{R} \times B^{h^{1/2}}_{0,\rho} \to B^{h^{1/2}}_{0,\rho},\ (t, (x^{(0)}, y^{(0)}) \mapsto \mathcal{S}^{h^{1/2}}_t (x^{(0)},\, y^{(0)}) \tag{9.29}$$

is continuous and for any $n \in \mathbb{Z}$

$$\partial_t x_n(t) = \omega_n y_n(t), \quad \partial_t y_n(t) = -\omega_n x_n(t), \quad x_n(0) = x^{(0)}_n, \quad y_n(0) = y^{(0)}_n.$$

Furthermore, for any $v^{(0)} \in V_{0,\rho}$ and $t \in \mathbb{R}$ define

$$\mathcal{S}_t(V^{(0)}) := \Phi^{-1}\big(\mathcal{S}^{h^{1/2}}_t (\Phi(v^{(0)}))\big).$$

By Proposition 9.28, $\mathcal{S}_t(v^{(0)}) \in \operatorname{Iso}(v^{(0)})$ since by (9.28) $\mathcal{S}^{h^{1/2}}_t (\Phi(v^{(0)}))$ is in $\operatorname{Tor}(\Phi(v^{(0)}))$. Clearly, the map

$$\mathbb{R} \times V_{0,\rho} \to V_{0,\rho},\ (t, v^{(0)}) \mapsto \mathcal{S}_t(v^{(0)})$$

is continuous. Corollary 9.32 then yields the following

***Theorem* 9.33** *Let $\mathcal{S}$ be any finite subset of $\mathbb{Z}$. Then for any $v^{(0)} \in \mathcal{G}_{\mathcal{S},\rho}$, $t \mapsto v(t) := \mathcal{S}_t(v^{(0)})$ is a C^1-curve with values in $\mathrm{Iso}(v^{(0)}) \subseteq V_{0,\rho}$ and a global in time solution of the* sinh*-Gordon equation*

$$\partial_t v(t) = JP^{-1}\partial H_{\mathrm{sinh}}(v(t)), \quad v(0) = v^{(0)}.$$

The Hamiltonian vector field $JP^{-1}\partial H_{\mathrm{sinh}}(v(t))$ is in H^1_r and $\partial H_{\mathrm{sinh}}(v(t))$ given by

$$\partial H_{\mathrm{sinh}}(v(t)) = \sum_{n\in\mathcal{S}} \omega_n \partial I_n(v(t))$$

where $\omega_n = \partial_{I_n} H$ is the n'th frquencey of the sinh*-Gordon Hamiltonian H, expressed in the action variables.*

Remark 9.34. Since $\bigcup_{\substack{\mathcal{S}\subseteq\mathbb{Z}\\ |\mathcal{S}|<\infty}} \mathcal{G}_{\mathcal{S},\rho}$ is dense in $V_{0,\rho}$ the map $\mathcal{S} : \mathbb{R}\times V_{o,\rho} \to V_{0,\rho}$, $(t, v^{(0)}) \mapsto \mathcal{S}_t(v^{(0)})$ is continuous, and for any $v^{(0)} \in \bigcup_{\substack{\mathcal{S}\subseteq\mathbb{Z}\\ |\mathcal{S}|<\infty}} \mathcal{G}_{\mathcal{S},\rho}$, the curve $t \mapsto \mathcal{S}_t(v^{(0)})$ is a solution of the sinh-Gordon equation. We refer to $\mathcal{S} : (t, v^{(0)}) \mapsto \mathcal{S}_t(v^{(0)})$ as the solution map of the sinh-Gordon equation.

Remark 9.35. The fact that the solutions in $V_{0,\rho}$ of the sinh-Gordon equation is a rotational flow, when expressed in Birkhoff coordinates, has many implications. In particular, arguing as in the proof of Proposition 4.6 in [10] it follows that any solution of the sinh-Gordon equation in $V_{0,\rho}$ is almost periodic.

Proof of Theorem 9.33. Since $\Phi : V_{0,\rho} \to B^{h^{1/2}}_{0,\rho}$ is a real analytic diffeomorphism and for any given $v^{(0)} \in \mathcal{G}_{\mathcal{S},\rho}$, the curve $t \mapsto v(t) := \mathcal{S}^{h^{1/2}}_t(\Phi(v^{(0)}))$ is clearly smooth in t, $t \mapsto \mathcal{S}_t(v^{(0)})$ is a C^1-curve with values in $V_{0,\rho}$. Furthermore, by Corollary 9.32, $\partial H_{\mathrm{sinh}}(v(t)) = \sum_{n\in\mathcal{S}} \omega_n \partial I_n(v(t))$. It therefore remains to show that for any $n \in \mathcal{S}$

$$JP^{-1}\partial I_n(v(t)) = (\mathrm{d}_{v(t)}\,\Phi)^{-1}\big((y_n(t)\delta_{nk})_k,\ (-x_n(t)\delta_{nk})_k\big). \tag{9.30}$$

Note that for any $n \in \mathcal{S}$, the vector $(y_n(t), -x_n(t)) \neq 0$. Since $x_n(t)^2 + y_n(t)^2 = 2I_n(t) = 2I_n(0) \neq 0$ by assumption. Therefore $-(y_n(t), -x_n(t))$ equals ∂_{θ_n} at the point considered. The claimed identity (9.30) then follows from Lemma 9.36 below. □

***Lemma* 9.36** *For any $v \in \mathcal{G}_{\mathcal{S},\rho}$ and for any $n \in \mathcal{S}$,*

$$\mathrm{d}_{\Phi(v)}\,\Phi^{-1}[\partial_{\theta_n}] = -JP^{-1}\partial I_n.$$

Proof. Choose ∂_{I_k}, ∂_{θ_k} $(k \in \mathcal{S})$ ∂_{x_k}, ∂_{y_k} $(k \in \mathbb{Z}\setminus\mathcal{S})$ as a basis of the tangent space $T_{\Phi(v)}g_{\mathcal{S},\rho}$ and define for any $n \in \mathcal{S}$, $e_n := \mathrm{d}_{\Phi(v)}\,\Phi^{-1}[\partial_{\theta_n}]$. When expressed in the chosen basis

$$\begin{aligned} \partial I_k[e_n] =&0, \quad \partial\theta_k[e_n] = \delta_{nk} \quad \forall\ k \in \mathcal{S}\\ \partial x_k[e_n] =&0, \quad \partial y_k[e_n] = 0 \quad \forall\ k \in \mathbb{Z}\setminus\mathcal{S}. \end{aligned}$$

On the other hand, the vector $-JP^{-1}\partial I_n$ has the following components: for any $k \in \mathcal{S}$ by Proposition 9.8 and Proposition 9.7

$$\begin{aligned} \partial I_k[-JP^{-1}\partial I_n] &= -\langle \partial I_k,\ JP^{-1}\partial I_n\rangle = -\{I_k,\ I_n\} = 0,\\ \partial\theta_k[-JP^{-1}\partial I_n] &= -\langle \partial\theta_k,\ JP^{-1}\partial I_n\rangle = -\{\theta_k,\ I_n\} = \delta_{nk} \end{aligned}$$

and for any $k \in \mathbb{Z}\setminus\mathcal{S}$ by Corollary 9.11,

$$\begin{aligned} \partial x_k[-JP^{-1}\partial I_n] &= -\ \{x_k,\ I_n\} = 0\\ \partial y_k[-JP^{-1}\partial I_n] &= -\ \{y_k,\ I_n\} = 0. \end{aligned}$$

Since $\mathrm{d}_v\,\Phi$ is a linear isomorphism it follows that $e_n = -JP^{-1}\partial I_n$ as claimed. □

10 Examples

10.1 $Pp + q_x = 0$ example

Let $s \geq 0$ and $q \in H^{s+1}_{\mathbb{R}}$ and define $p = -P^{-1}q_x \in H^{s+1}_{\mathbb{R}}$. Then $A(q,p) = 0$ and the fundamental solution satisfies

$$\partial_x M = J(\lambda - B^2/\lambda)M = \begin{pmatrix} & \lambda - \frac{1}{16\,\lambda}e^{q} \\ -\lambda + \frac{1}{16\,\lambda}e^{-q} & \end{pmatrix} M =: YM$$

Hence $M(x,\lambda,q,p) = e^{\int_0^x Y(s,\lambda,q,p)\,\mathrm{d}s}$. Let $a^{\pm}$ be the eigenvalues of $\int_0^1 Y(s,\lambda,q,p)\,\mathrm{d}s$ then

$$\Delta(\lambda,q,p) = \frac{\operatorname{tr}\grave{M}}{2} = \frac{e^{a^+} + e^{a^-}}{2}.$$

Since the eigenvalues a^+, a^- of

$$\int_0^1 Y(s,\lambda,q,p)\,\mathrm{d}s = \begin{pmatrix} & \lambda - \frac{1}{16\,\lambda}\int_0^1 e^{q(s)}\,\mathrm{d}s \\ -\lambda + \frac{1}{16\,\lambda}\int_0^1 e^{-q(s)}\,\mathrm{d}s & \end{pmatrix}$$

are given by $a^2 - \operatorname{tr}\Big(\int_0^1 Y(s,\lambda,q,p)\,\mathrm{d}s\Big)a + \det\Big(\int_0^1 Y(s,\lambda,q,p)\,\mathrm{d}s\Big) = 0$, one has

$$\begin{aligned} a^{\pm}(\lambda,q,p) &= \pm\sqrt{\left(\lambda - \frac{1}{16\,\lambda}\int_0^1 e^{q(s)}\,\mathrm{d}s\right)\left(-\lambda + \frac{1}{16\,\lambda}\int_0^1 e^{-q(s)}\,\mathrm{d}s\right)} \\ &= \pm\sqrt{-\lambda^2 + \frac{1}{8}\int_0^1 \cosh(q(s))\,\mathrm{d}s - \frac{1}{(16\,\lambda)^2}\int_0^1 e^{q(s)}\,\mathrm{d}s\int_0^1 e^{-q(s)}\,\mathrm{d}s}. \end{aligned}$$

Where the sign of the root is immaterial. Hence

$$\Delta(\lambda,q,p) = \cosh\left(\sqrt{-\lambda^2 + \frac{1}{8}\int_0^1 \cosh(q(s))\,\mathrm{d}s - \frac{1}{(16\,\lambda)^2}\int_0^1 e^{q(s)}\,\mathrm{d}s\int_0^1 e^{-q(s)}\,\mathrm{d}s}\right).$$

Hence $\lambda \in \mathbb{C}^*$ is a periodic eigenvalue of $Q(q,p)$ if and only if there is some $n \in \mathbb{Z}$ such that

$$\lambda^2 + \frac{1}{(16\,\lambda)^2}\int_0^1 e^{q(s)}\,\mathrm{d}s\int_0^1 e^{-q(s)}\,\mathrm{d}s = (n\pi)^2 + \frac{1}{8}\int_0^1 \cosh(q(s))\,\mathrm{d}s \tag{10.1}$$

For $n \neq 0$ and λ_n a solution of (10.1) one hence has

$$\dot\Delta(\lambda_n,q,p) = \sinh(\pm in\pi)\frac{-\lambda_n + \frac{1}{(16)^2\lambda_n^3}\int_0^1 e^{q(s)}\,\mathrm{d}s\int_0^1 e^{-q(s)}\,\mathrm{d}s}{\pm in\pi} = 0$$

implying that every periodic eigenvalue except $\lambda_0^{\pm}$ is double, i.e. $\gamma_n = 0$ for all $n \neq 0$. We now have proved

***Proposition* 10.1** *For any $s \geq 0$ there are $(q,p) \in H^{s+1}_c$ such that $(q,p) \notin H^{s+1+\epsilon}_c$ for any $\epsilon > 0$ such that $\gamma_n(q,p) = 0$ for all $n \neq 0$.*

One can further compute with $c_1 := \int_0^1 \cosh(q(s))\,\mathrm{d}s$ and $c_2 := \int_0^1 e^{q(s)}\,\mathrm{d}s\int_0^1 e^{-q(s)}\,\mathrm{d}s$ for $n \neq 0$

$$\lambda_n^{\pm} = -1/4\sqrt{8\pi^2n^2 + c_2 + \operatorname{sign}(n)\sqrt{64\pi^4n^4 + 16\pi^2c_2n^2 + c_2^2 - c_1}}$$

and

$$\lambda_0^{\pm} = -1/4\sqrt{c_2 \pm \sqrt{c_2^2 - c_1}}$$

11 Appendices

A Analytic maps

For the convenience of the reader we recall in this appendix the notion of an analytic map between two complex Banach spaces and record well known results about such maps used throughout the thesis. The material is taken almost verbatim from [[6], Appendix A].

Let E and F be complex Banach spaces with norms $|\cdot|$ and $\|\cdot\|$, respectively, and let $U \subset E$ be open. A map $f : U \to F$ is analytic on U, if it is differentiable on U. This is the straightforward generalization of the notion of an analytic function of one complex variable.

It is convenient to introduce another notion of analyticity. A map $f : U \to F$ is weakly analytic on U, if for each $U \in U$, $h \in E$, and $L \in F^*$, the function

$$z \mapsto Lf(u + zh)$$

is analytic in some neighborhood of the origin in $\mathbb{C}$ in the usual sense. The radius of weak analyticity of f at u is the supremum of all $r \geq 0$ such that the above function is defined and analytic in the disc $z < r/|h|$ (in $\mathbb{C}$) for all $L \in F^*$ and $h \in E$.

Clearly, the radius r of weak analyticity at u is not greater than the distance ρ of u to the boundary of U. On the other hand, if L and h are given, then $z \mapsto Lf(u+zh)$ is well defined on the disc $|z| < r|h|$ (in $\mathbb{C}$) and analytic in some neighborhood of each point in it, since f is weakly analytic on all of U. Consequently, this function is analytic on $|z| < \rho/|h|$.

The notion of a weak analytic map is weaker then that of an analytic map. For instance, every globally defined, but unbounded linear operator is weakly analytic, but not analytic. Remarkably, a weakly analytic map is analytic, if in addition it is locally bounded. Before we get to this result, we state two basic lemmas. For proofs we refer to [6]

***Lemma* A.1** (Cauchy's Formula) *Suppose f is weakly analytic and continuous on U. Then, for every $u \in U$ and $h \in E$,*

$$f(u + zh) = \frac{1}{2\pi \mathrm{i}} \int_{|\zeta|=\rho} \frac{f(u + \zeta h)}{\zeta - z} \mathrm{d}\zeta$$

for $|z| < \rho < r/|h|$, where r is the radius of weak analyticity of f at u.

***Lemma* A.2** (Cauchy's Estimate) *Let f be an analytic map from the open ball of radius r around u in E onto F such that $\|f\| \leq M$ on this ball. Then*

$$\|d_u f\| := \max_{h \neq 0} \frac{\|d_u f(h)\|}{|h|} \leq \frac{M}{r}.$$

The statement of the lemma is particularly transparent when f is a complex valued function. Then $d_u f$ is an element in the dual space E^* to E, and the induced operator norm is the norm $|\cdot|_E$. So, for instance, if f is bounded in absolute value by M on the balls

$$|u|_\infty,\ |u|_2,\ |u|_1\ <\ r,$$

then

$$|d_0 f|_1,\ |d_0 f|_2,\ |d_0 f|_\infty\ \leq\ \frac{M}{r},$$

respectively, in both finite- and infinite-dimensional settings.

We now turn to the basic characterization of analytic maps between complex Banach spaces. An infinitely differentiable function f is said to be represented by its Taylor series near a point u, if

$$f(u + h) = \sum_{n \geq 0} \frac{1}{n!} d_u^n f(h, \ldots, h),$$

for all sufficiently small h, with the series converging absolutely and uniformly.

For example, since $E = H^1(\mathbb{T}, \mathbb{C})$ is a Banach algebra by the Sobolev embedding theorem, any map

$$P : E \to E, \quad \varphi \mapsto \sum_{n=0}^{m} a_n \varphi^n$$

with coefficients $a_0, \ldots, a_m \in E$ is well defined, analytic, and represented by its Taylor series at any point. In particular,

$$\frac{1}{n!} d_0^n P(h, \ldots, h) = a_n h^n, \quad 0 \le n \le m.$$

Similarly, the exponential series

$$\exp : E \to E, \quad \exp(\varphi) = \sum_{n \ge 0} \frac{\varphi^n}{n!}$$

is well defined and analytic, with

$$d_0^n \exp(h, \ldots, h) = h^n, \quad n \ge 0.$$

Note that $(\exp(\varphi))(x) = \exp(\varphi(x))$.

***Theorem* A.3** *Let $f : U \to F$ be a map from an open subset U of a complex Banach space E into a complex Banach space F. Then the following three statements are equivalent.*

(i) *f is weakly analytic and locally bounded on U.*

(ii) *f is analytic on U.*

(iii) *f is infinitely differentiable on U, and is represented by its Taylor series in a neighborhood of each point in U.*

Theorem A.3 allows us to generalize results about analytic functions of one complex variable to the setting of this section. For instance, the following holds.

***Theorem* A.4** *Let $(f_n)_n$ be a sequence of analytic maps $f_n : U \to F$ from an open subset U of a complex Banach space E into a complex Banach space F, which converges locally uniformly to a map $f : U \to F$. Then f is analytic as well.*

A special case of Theorem A.3 arises for maps into a Hilbert space.

***Theorem* A.5** *Let $f : U \to H$ be a map from an open subset U of a complex Banach space E into a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and an orthonormal basis $(e_n)_n$. Then f is analyttic on U if and only if f is locally bounded, and each coordinate function*

$$f_n = \langle f, e_n \rangle : U \to \mathbb{C}, \quad n \in \mathbb{Z},$$

is analytic on U. Moreover, the derivative of f is given by

$$df(h) = \sum_{n \in \mathbb{Z}} df_n(h) e_n.$$

The next theorem generalizes Theorem A.3. A subset $V \subset U$ of an open set U in a complex Banach space is called an analytic subvariety, if locally it can be represented as the zero set of an analytic function taking values in some $\mathbb{C}^n$.

***Theorem* A.6** *Let $V_1, \ldots, V_m$ be analytic subvarieties of an open subset U in a complex Banach space E. Suppose the function $f : U \to \mathbb{C}$ is*

(i) *analytic on $U \setminus (V_1 \cup \cdots \cup V_m)$,*

(ii) *continuous on U, and,*

(iii) *weakly analytic on V_i for each $q \le i \le m$, that is, analytic on any complex disc contained in V_i.*

then f is analytic on U.

***Theorem* A.7** *Let V be a real subspace of complex Banach space E and U an open neighborhood. Let $f_n : U \to \mathbb{C}$ be analytic such that the sum $\sum_{n \ge 0} |f_n|$ converges locally uniformly on V then there is an open neighborhood U' of $V \subset U' \subset U$ such that $\sum_{n \ge 0} f_n$ converges to an analytic function on U'.*

Proof. By assumption $F = \sum_{n\geq 0} f_n\Big|_V$ is a real analytic function. Let $z_0 \in V$ then F has a Taylor expansion $F(\lambda) = \sum_{k\geq 0} a_k(z-z_0)^k$ around λ_0. By Fubinis theorem we obtain

$$a_k = \frac{1}{2\pi \mathrm{i}} \int_C \frac{F(z)}{(z-z_0)^{k+1}}\,\mathrm{d}z = \sum_{n\geq 0} \frac{1}{2\pi \mathrm{i}} \int_C \frac{f_n(z)}{(z-z_0)^{k+1}}\,\mathrm{d}z = \sum_{n\geq 0} a_k^n$$

Where C is a curve in $\mathbb{C}$ around Further more by assumption we have

$$\sum_{n\geq 0} |a_k^n| \leq \sum_{n\geq 0} \frac{1}{2\pi \mathrm{i}} \int_C \frac{|f_n(z)|}{|z-z_0|^{k+1}}\,\mathrm{d}z$$

□

B Infinite products

In this appendix we record for the convenience of the reader well known results on infinite products used throughout the thesis. The material is taken almost verbatim from [[6], Appendix C]

We say that the infinite product $\prod_{n\in\mathbb{Z}}(1+a_n)$ of complex numbers a_n, $n \in \mathbb{Z}$ converges, if

$$\lim_{N\to\infty} \prod_{|n|\leq N} (1+a_n)$$

exists. The limit is also denoted by $\prod_{n\in\mathbb{Z}}(1+a_n)$. We say that $\prod_{n\in\mathbb{Z}}(1+a_n)$ converges absolutely if $\prod_{n\in\mathbb{Z}}(1+|a_n|)$ converges. Absolute convergence of an infinite product implies its convergence. A sufficient condition for absolute convergence is that

$$|a|_{\ell^1} := \sum_{n\in\mathbb{Z}} |a_n| < \infty,$$

since for sufficiently large N,

$$\sum_{|n|>N} \log(1+|a_n|) \leq \sum_{n\in\mathbb{Z}} |a_n|.$$

This condition is also necessary, since

$$\sum_{|n|>N} \log(1+|a_n|) \geq \sum_{|n|>N} |a_n|(1-|a_n|) \geq \frac{1}{2} \sum_{|n|>N} |a_n|$$

for N sufficiently large. A sufficient condition for convergence is

$$\sum_{n\geq 1}(|a_n + a_{-n}| + |a_n a_{-n}|) < \infty,$$

obtained by grouping the factors $1+a_n$ and $1+a_{-n}$. Often we are concerned with infinite products related to perturbations of the sine function, which has the following product representation

$$\sin\lambda = -\prod_{m\in\mathbb{Z}} \frac{m\pi - \lambda}{\pi_m}, \quad \pi_m = \begin{cases} m\pi & m \neq 0 \\ 1 & m = 0 \end{cases}.$$

We consider perturbations obtained by replacing the sequence $\sigma^0 = (m\pi)_{m\in\mathbb{Z}}$ of zeros of $\sin\lambda$ by a sequence of the form $\sigma = \sigma^0 + \tilde{\sigma}$ with $\tilde{\sigma} \in \ell^2_{\mathbb{C}}$. We say that such a sequence σ is simple if its elements are pairwise distinct. Note that

$$\inf_{m\neq n} |\sigma_m - \sigma_n| > 0$$

for any simple sequence. We will only consider sequences where $\sigma_m \neq 0$ for all m. Define

$$\ell^* := \{\ \sigma = (\sigma_n)_n \subset \mathbb{C}^* \ :\ \sigma - \sigma^0 \in \ell^2\ \}$$

***Lemma* B.1** *For any $\sigma \in \ell^*$ and $n \in \mathbb{Z}$,*

$$f_n(\lambda, \sigma) = -\prod_{m \neq n} \frac{\sigma_m - \lambda}{\pi_m}$$

defines an analytic function on $\mathbb{C} \times \ell^*$ with roots σ_m, $m \neq n$, listed with their multiplicities. In particular, if σ is simple, then f_n has simple roots σ_m, $m \neq n$, and no other roots.

To obtain asymptotic estimates for the infinite products considered in Lemma B.1 we need the following general inequality

***Lemma* B.2** *For any ℓ^1-sequence of complex numbers a_m with $|a_m| \leq 1/2$,*

$$\left| \prod_{m \in \mathbb{Z}} (1 + a_m) - 1 \right| \leq Ae^S + Be^{S+S^2}$$

with $A = |\sum_{m \in \mathbb{Z}} a_m|$, $B = \sum_{m \in \mathbb{Z}} |a_m|^2$, and $S = \sum_{m \in \mathbb{Z}} |a_m|$.

Let

$$D_n = \{\, \lambda \in \mathbb{C} : \; |\lambda - n\pi| < \pi/4 \,\}, \quad n \in \mathbb{Z}.$$

For any $a = (a_n)_n$ define $\|a\|_{\ell^\infty} := \sup_{n \in \mathbb{Z}} |a_n|$.

***Lemma* B.3** *Let $\sigma, \rho \in \ell^*$ be two complex sequences and assume that ρ is simple. Furthermore, let $N \geq 0$ be an integer and $c > 0$ a constant such that for any n with $|n| \geq N$*

$$\min_{\lambda \in D_n} |\rho_m - \lambda| \geq c^{-1} |m - n|, \qquad \forall\, m \in \mathbb{Z} \setminus \{n\}$$

Then for any $n \in \mathbb{Z}$ with $|n| \geq N$

$$\prod_{m \neq n} \frac{\sigma_m - \lambda}{\rho_m - \lambda} = 1 + \ell_n^2, \quad \lambda \in \mathrm{D}_n,$$

uniformly with respect to $\|\sigma - \rho\|_{\ell^2}$, $\|\rho - \sigma^0\|_{\ell^\infty}$, and c. In more detail, the infinite product $\phi_n(\lambda) = \prod_{m \neq n} \frac{\sigma_m - \lambda}{\rho_m - \lambda}$ satisfies

$$\Big(\sum_{|n| \geq N} \sup_{\lambda \in D_n} |\phi_n(\lambda) - 1|^2 \Big)^{1/2} < C_{\sigma,\rho}$$

where the constant $C_{\sigma,\rho}$ only depends on $\|\sigma - \rho\|_{\ell^2}$, $\|\rho - \sigma^0\|_{\ell^\infty}$, and c.

Lemma B.3 can be used to prove the following

***Lemma* B.4** *For any $(\lambda_n)_n \in \ell^*$*

$$\frac{1}{\pi_n} \prod_{m \neq n} \frac{\lambda_m - \lambda}{\pi_m} = \frac{\sin(\lambda)}{\lambda - n\pi} \left(1 + \ell_n^2\right), \quad \lambda \in D_n,$$

locally uniformly in $(\lambda_n)_n$. We note that $\frac{n\pi - \lambda}{\sin(\lambda)}$ has a removable singularity at $n\pi$, but no other singularity in D_n. Furthermore for any $\lambda \in D_n$

$$\frac{n\pi - \lambda}{\sin(\lambda)} \prod_{m \neq n} \frac{\lambda_m - \lambda}{\pi_m} = -\pi_n \prod_{m \neq n} \frac{\lambda_m - \lambda}{m\pi - \lambda}.$$

We refer to Lemma B.3 for the meaning of the statement on uniformity.

We also need an estimate of infinite products on large circles.

***Lemma* B.5** *For any $(\sigma_m)_{m \in \mathbb{Z}} \in \ell^*$, the infinite product*

$$f(\lambda) := \prod_{m \in \mathbb{Z}} \frac{\sigma_m - \lambda}{\pi_m}$$

defines an entire function with roots σ_m, $m \in \mathbb{Z}$. On the circles $\partial B_N = \{\, \lambda \in \mathbb{C} : \; |\lambda| = N\pi + \pi/2 \,\}$, it satisfies

$$f(\lambda) = -(1 + o(1)) \sin(\lambda) \quad \text{as} \quad N \to \infty.$$

In case $\lambda_m \neq 0 \;\; \forall\, m \in \mathbb{Z}$, f can be written in the form

$$f(\lambda) = f(0) \prod_{m \in \mathbb{Z}} \left(1 - \frac{\lambda}{\lambda_m}\right), \quad f(0) = \prod_{m \in \mathbb{Z}} \frac{\lambda_m}{\pi_m}.$$

C A version of Liouville's theorem on $\mathbb{C}^*$

Recall that at the beginning of Chapter 3 we have introduced the discs B_n, $n \in \mathbb{Z}$, where $B_0 = \{ \lambda : |\lambda| < \pi/2 \}$ and, for any $n \geq 1$,

$$B_n = \{ \lambda : |\lambda| < n\pi + \pi/2 \}, \quad B_{-n} = \{ \lambda : |\lambda| \leq \frac{1}{16(n\pi + \pi/2)} \}.$$

***Lemma* C.1** *Let f be an analytic function on $\mathbb{C}^*$ which is uniformly bounded on the circles ∂B_n and ∂B_{-n} as $n \to \infty$. Then f is constant.*

Proof. First note that if f is bounded by $M > 0$ on all these circles then so is g, defined by $g(\lambda) = f\left(\frac{1}{16\,\lambda}\right)$. Furthermore if the Laurent series of f around 0 is given by

$$f(\lambda) = \sum_{k \in \mathbb{Z}} a_k \lambda^k$$

then the Laurent series of g is given by

$$g(\lambda) = \sum_{k \in \mathbb{Z}} a_{-k} (16\lambda)^k.$$

Hence by Cauchy's inequality $|a_k| \leq r^{-k} \sup_{|z|=r} |f(z)|$ for any $r > 0$. Choosing $r = n\pi + \pi/2$ we can bound $|f(z)|$ by M and hence $a_k = 0$ for any $k \geq 1$. Using g, the same argument shows that $a_{-k} = 0$ for any $k \geq 1$. Hence $f = g \equiv a_0$ is constant. □

D Interpolation

In this appendix we present an interpolation lemma which will be used to construct the ψ-functions and to verify canonical relations. Recall that for any $N \geq 1$, we denote by $A_N = B_N \setminus B_{-N}$ the annulus, centered at 0, with boundary $\partial A_N = \partial B_N \cup \partial B_{-N}$ where $B_N = \{ \lambda : |\lambda| < N\pi + \pi/2 \}$ and $B_{-N} = \{ \lambda : |\lambda| \leq (16(N\pi + \pi/2))^{-1} \}$. Assume that $(\sigma_{1,n})_{n \in \mathbb{Z}}$, $(\sigma_{2,n})_{n \in \mathbb{Z}}$ are sequences in $\mathbb{C}^*$ so that, with $\kappa_{2,n} := (-16\sigma_{2,n})^{-1}$

$$\sigma_{1,n} \neq \sigma_{1,m} \ (n \neq m), \quad \kappa_{2,n} \neq \kappa_{2,m} \ (m \neq n), \quad \sigma_{1,n} \neq \kappa_{2,m} \ (n, m \in \mathbb{Z}) \tag{D.1}$$

and

$$\sigma_{1,n}, \sigma_{2,n} = n\pi + \ell_n^2 \quad \text{as } n \to \infty. \tag{D.2}$$

Let

$$f(\lambda) = f_1(\lambda) f_2(\lambda) \tag{D.3}$$

where

$$f_1(\lambda) := \prod_{n \in \mathbb{Z}} \frac{\sigma_{1,n} - \lambda}{\pi_n}, \quad f_2(\lambda) := \prod_{n \in \mathbb{Z}} \frac{\sigma_{2,n} + \frac{1}{16\lambda}}{\pi_n}. \tag{D.4}$$

According to Lemma B.1, the infinite products f_1, f_2 define analytic functions on $\mathbb{C}^*$ with roots $\sigma_{1,n}$, $n \in \mathbb{Z}$, and respectively $\kappa_{2,n}$, $n \in \mathbb{Z}$. Note that $f_1(\lambda)$ is analytic at 0 with $f_1(0) \neq 0$ and $f_2(\lambda)$ is analytic at ∞ with $f_2(\infty) \neq 0$,

$$f_1(0) = \prod_{n \in \mathbb{Z}} \frac{\sigma_{1,n}}{\pi_n}, \quad f_2(\infty) = \prod_{n \in \mathbb{Z}} \frac{\sigma_{2,n}}{\pi_n}.$$

Furthermore, by Lemma B.5

$$\sup_{\lambda \in \partial B_N} \Big| \frac{f_1(\lambda)}{\sin(\lambda)} + 1 \Big| = o(1) \quad \text{as } N \to \infty \tag{D.5}$$

and

$$\sup_{\lambda \in \partial B_{-N}} \Big| \frac{f_2(\lambda)}{\sin(-\frac{1}{16\lambda})} + 1 \Big| = \sup_{\mu \in \partial B_N} \Big| \frac{f_2(-\frac{1}{16\mu})}{\sin(\mu)} + 1 \Big| = o(1) \quad \text{as } N \to \infty \tag{D.6}$$

***Lemma* D.1** *Assume that* $\varphi : \mathbb{C}^* \to \mathbb{C}$ *is analytic and that* $(\sigma_{1,n})_{n\in\mathbb{Z}}$, $(\sigma_{2,n})_{n\in\mathbb{Z}}$ *are sequences in* $\mathbb{C}^*$ *so that* (D.1)-(D.2) *are satisfied. Furthermore assume that as* $N \to \infty$

$$\sup_{\lambda\in\partial B_N} |\frac{\varphi(\lambda)}{\sin(\lambda)}| = o(1), \quad \sup_{\lambda\in\partial B_{-N}} |\frac{\varphi(\lambda)}{\sin(-\frac{1}{16\lambda})}| \ \Big(= \sup_{\mu\in\partial B_N} |\frac{\varphi(-\frac{1}{16\mu})}{\sin(\mu)}|\Big) = o(N). \tag{D.7}$$

Then for any $z \in \mathbb{C}^*$

$$\varphi(z) = \sum_{n\in\mathbb{Z}} \Big(\frac{\varphi(\sigma_{1,n})}{\dot f(\sigma_{1,n})} \frac{f(z)}{z-\sigma_{1,n}} + \frac{\varphi(\kappa_{2,n})}{\dot f(\kappa_{2,n})} \frac{f(z)}{z-\kappa_{2,n}}\Big)$$

where f *is the function defined in* (D.3), $\dot f(\lambda) = \partial_\lambda f(\lambda)$, *and* $\kappa_{2,n} = -(16\sigma_{2,n})^{-1}$, $n \in \mathbb{Z}$. *In particular, the latter sum converges.*

Proof. Consider for any $z \in \mathbb{C}^* \setminus \{ \sigma_{1,n}, \kappa_{2,n} : n \in \mathbb{Z} \}$ the function

$$g(\lambda) := \frac{\varphi(\lambda)}{(\lambda - z) f(\lambda)}$$

where $f(\lambda)$ is given by (D.3). Then $g(\lambda)$ is a meromorphic function on $\mathbb{C}^*$ with poles in z, $\sigma_{1,n}$ ($n \in \mathbb{Z}$), $\kappa_{2,n}$ ($n \in \mathbb{Z}$). Chose $N \geq 1$ so large that,

$$z \in A_N, \quad \sigma_{1,n}, \kappa_{2,n} \in A_N \ \ \forall\, |n| \leq N, \quad \sigma_{1,n}, \kappa_{2,n} \in \mathbb{C}^* \setminus A_N \ \ \forall\, |n| \geq N+1.$$

Then by the residue theorem

$$\begin{aligned} \frac{1}{2\pi \mathrm{i}} \int_{\partial A_N} g(\lambda)\, \mathrm{d}\lambda =& \operatorname{Res}_z g + \sum_{|n|\leq N} \operatorname{Res}_{\sigma_{1,n}} g + \sum_{|n|\leq N} \operatorname{Res}_{\kappa_{2,n}} g \\ =& \frac{\varphi(z)}{f(z)} + \sum_{|n|\leq N} \frac{\varphi(\sigma_{1,n})}{\dot f(\sigma_{1,n})} \frac{1}{\sigma_{1,n} - z} + \sum_{|n|\leq N} \frac{\varphi(\kappa_{2,n})}{\dot f(\kappa_{2,n})} \frac{1}{\kappa_{2,n} - z} \end{aligned}$$

On the other hand, write $\frac{1}{2\pi \mathrm{i}} \int_{\partial A_N} g(\lambda)\, \mathrm{d}\lambda = \frac{1}{2\pi \mathrm{i}} I_N - \frac{1}{2\pi \mathrm{i}} II_N$ where

$$I_N = \int_{\partial B_N} \frac{\varphi(\lambda)}{f(\lambda)} \frac{1}{\lambda - z}\, \mathrm{d}\lambda, \quad II_N = \int_{\partial B_{-N}} \frac{\varphi(\lambda)}{f(\lambda)} \frac{1}{\lambda - z}\, \mathrm{d}\lambda.$$

It follows from (D.5), (D.7) that

$$\sup_{\lambda\in\partial B_N} \Big|\frac{\varphi(\lambda)}{f(\lambda)}\Big| = o(1) \quad \text{as } N \to \infty$$

implying that $\lim_{N\to\infty} I_N = 0$. Similarly it follows from (D.6), (D.7) that

$$\sup_{\lambda\in\partial B_{-N}} \Big|\frac{\varphi(\lambda)}{f(\lambda)}\Big| = \sup_{\mu\in\partial B_N} \Big|\frac{\varphi(-\frac{1}{16\mu})}{f(-\frac{1}{16\mu})}\Big| = o(N) \quad \text{as } N \to \infty$$

yielding that

$$\int_{\partial B_{-N}} \frac{\varphi(\lambda)}{f(\lambda)} \frac{1}{\lambda - z}\, \mathrm{d}\lambda = \int_{\partial B_N} \frac{\varphi(-\frac{1}{16\mu})}{f(-\frac{1}{16\mu})} \frac{1}{-\frac{1}{16\mu} - z} \frac{d\mu}{16\mu^2} \to 0 \quad \text{as } N \to \infty.$$

Combining the results obtained one gets

$$0 = \frac{\varphi(z)}{f(z)} + \sum_{n\in\mathbb{Z}} \Big(\frac{\varphi(\sigma_{1,n})}{\dot f(\sigma_{1,n})} \frac{1}{\sigma_{1,n} - z} + \frac{\varphi(\kappa_{2,n})}{\dot f(\kappa_{2,n})} \frac{1}{\kappa_{2,n} - z}\Big)$$

and hence

$$\varphi(z) = \sum_{n\in\mathbb{Z}} \Big(\frac{\varphi(\sigma_{1,n})}{\dot f(\sigma_{1,n})} \frac{f(z)}{z-\sigma_{1,n}} + \frac{\varphi(\kappa_{2,n})}{\dot f(\kappa_{2,n})} \frac{f(z)}{z-\kappa_{2,n}}\Big). \tag{D.8}$$

Clearly $\frac{f(z)}{z-\sigma_{1,n}}$ has a removable singularity at $z = \sigma_{1,n}$ and $\frac{f(z)}{z-\kappa_{2,n}}$ has one at $z = \kappa_{2,n}$. This implies that the identity (D.8) actually holds for any $z \in \mathbb{C}^*$. □

E Perturbed Fourier transform

In this appendix, for the convenience of the reader we record two well know results on perturbed Fourier coefficients which can be found e.g. in [6].

***Lemma* E.1** *Let $f \in L^2_{\mathbb{C}}([0,1])$. Then for every $\epsilon > 0$ there exists a $\lambda_* > 0$ such that*

$$\left| \int_0^x e^{\mathrm{i}\lambda(x-2s)} f(s)\,\mathrm{d}s \right| < \epsilon e^{|\mathrm{Im}\lambda| x}, \qquad 0 \le x \le 1,$$

for all $|\lambda| > \lambda_*$. This estimate also holds on a small neighborhood around f.

***Lemma* E.2** *Let $f \in L^2_{\mathbb{C}}([0,1])$, and let*

$$\phi_n(x) = \int_0^x e^{\mathrm{i}\xi_n(x-2s)} f(s)\,\mathrm{d}s, \qquad n \in \mathbb{Z},$$

with a complex sequence $\xi_n = n\pi + \alpha_n$ such that $a = \sup_n |\alpha_n| < \infty$. Then

$$\sum_{n\in\mathbb{Z}} |\phi_n(x)|^2 \le e^{2a} \|f\|^2, \quad 0 \le x \le 1.$$

Notation

Sets and spaces

$H^s_{\mathbb{C}} = H^s(\mathbb{T},\mathbb{C})$	$H^s_{\mathbb{R}} = H^s(\mathbb{T},\mathbb{R})$
$H^s_c = H^s_{\mathbb{C}} \times H^s_{\mathbb{C}},\ L^2_c \equiv H^0_c$	$H^s_r = H^s_{\mathbb{R}} \times H^s_{\mathbb{R}},\ L^2_r \equiv H^0_r$
$\ell^2 = \ell^2(\mathbb{Z},\mathbb{C})$	$\ell^2_{\mathbb{R}} = \ell^2(\mathbb{Z},\mathbb{R})$
$\ell^2_c = \ell^2 \times \ell^2$	W complex nbhd of H^1_r
$\hat{W}$ defined by (6.4)	U_n isolating nbhood around G_n
U_* isolating nbhood around $\dot{\lambda}_*$	$D_{-n} = \{\ \lambda \in \mathbb{C} :\ \dfrac{1}{16\lambda} \in D_n\ \}\ \ \forall\ n \geq 1$
Γ_n circuit around G_n in U_n	$D_0 = \{\ z \in \mathbb{C} :\ \lvert z - \dfrac{1}{4}\rvert < \dfrac{1}{4\pi}\ \}$
$D_n = \{\ \lambda \in \mathbb{C} :\ \lvert\lambda - n\pi < \pi/3\ \}\ \ \forall\ n \geq 1$	$B_0 = \{\ \lambda \in \mathbb{C} :\ \lvert\lambda - n\pi\rvert < \pi/3\ \}$
$B_n = \{\ \lambda \in \mathbb{C} :\ \lvert\lambda\rvert < n\pi + \pi/2\ \}\ \ \forall\ n \geq 1$	$B_{-n} = \{\ \lambda \in \mathbb{C} :\ \lvert\lambda\rvert \leq \dfrac{1}{16(n\pi+\pi/2)}\ \}\ \ \forall\ n \geq 1$
$A_n = B_n \setminus B_{-n}$	$Z_n = \{\ v \in \hat{W} :\ \gamma_n = 0\ \}$
$E_n = \{\ v \in W :\ \mu_n = \lambda^\pm_n\ \}$	$Y_n = \{\ v \in W :\ \mu_n \notin G_n\ \}$
$Iso(v_0) = \{\ v \in H^1_r :\ \Delta(\cdot,v) = \Delta(\cdot,v_0)\ \}$	

Spectral quantities

$\lambda^\pm_n$ periodic eigenvalues	$G_n = [\lambda^-_n, \lambda^+_n]$
μ_n Dirichlet eigenvalues	$\gamma_n = \lambda^+_n - \lambda^-_n\ \ \forall\ n \geq 0$
ν_n Neumann eigenvalues	$\gamma_{-n}(q,p) = \gamma_n(-q,p)\ \ \forall\ n \geq 1$
$\dot{\lambda}_n,\ \dot{\lambda}_*$ roots of $\dot{\Delta}$	$\Delta = (\grave{m}_1 + \grave{m}_4)/2$ discriminant
$\tau_n = (\lambda^+_n + \lambda^-_n)/2$	$\delta = (\grave{m}_1 - \grave{m}_4)/2$ anti - discriminant
$\chi_p = \Delta^2(\lambda) - 1$	$\chi_D = \grave{m}_2$
$\omega(\lambda) = \lambda - \dfrac{1}{16\lambda}$	
$\chi_1(\lambda) \equiv \chi_{p,1}(\lambda) = \displaystyle\prod_{n\in\mathbb{Z}} \frac{(\lambda^+_{n,1} - \lambda)(\lambda^-_{1,n} - \lambda)}{\pi^2_n}$	$\chi_2(\lambda) \equiv \chi_{p,2}(\lambda) = \displaystyle\prod_{n\in\mathbb{Z}} \frac{(\lambda^+_{n,2} + \frac{1}{16\,\lambda})(\lambda^-_{2,n} + \frac{1}{16\,\lambda})}{\pi^2_n}$

Roots

$\sqrt[+]{z}, z^{1/2}$ principal branch of square root on $\mathbb{C} \setminus (-\infty, 0]$

$$\sqrt[c]{\chi_1(\lambda)} = \prod_{k\in\mathbb{Z}} \frac{w_{1,k}(\lambda)}{\pi_k}$$

$$w_n(\lambda) = \sqrt[s]{(\lambda^+_n - \lambda)(\lambda^-_n - \lambda)}\ \ \forall\ n \geq 0$$

$$\sqrt[c]{\Delta^2(\lambda,q,p)} = \mathrm{i}\,\frac{\sqrt[c]{\chi_1(\lambda,q,p)}}{\sqrt[c]{\chi_1(0,q,p)}}\,\sqrt[c]{\chi_1(-(16\lambda)^{-1}, -q, p)}$$

Miscellaneous

$$\grave{m}_i = m_i\big|_{x=1} \quad 1 \le i \le 4$$

$$v = (q,p)$$

$$\varphi = \frac{1}{\mathrm{i}}(Pp + q_x)$$

$$\psi = \frac{1}{4}(Pp + q_x)$$

$$\xi_n = \sqrt[+]{I_n \tau_n/\gamma_n^2} \quad \forall\, n \ge 0$$

$$\pi_n = \begin{cases} n\pi, & \text{if } n \neq 0, \\ 1, & \text{if } n = 0. \end{cases}$$

$$\omega(\lambda) = \lambda - \frac{1}{16\,\lambda}$$

$$\langle n \rangle = \sqrt{1 + n^2\pi^2}$$

Operations

$\mathrm{d}F$ differential of F

$$\dot{F} = \partial_\lambda F$$

∂F L^2-gradient of F

$$\{G,F\} = -\int_0^1 \partial G J P^{-1} \partial F \,\mathrm{d}x$$

$$[\![f]\!]_{q,\lambda} = -\left(\frac{\lambda}{2} f \star f + \frac{1}{32\,\lambda} f \begin{pmatrix} e^{-q} & \\ & -e^{q} \end{pmatrix} f \atop \frac{1}{4} f \cdot Z f P(\cdot) \right)$$

$$\begin{pmatrix} a \\ b \end{pmatrix} \star \begin{pmatrix} c \\ d \end{pmatrix} = ac - bd.$$

$$\begin{pmatrix} a \\ b \end{pmatrix}^{\perp} = (-b, a)$$

$$\langle f, g \rangle_r = \int_0^1 f(x) g(x) \,\mathrm{d}x \quad \text{(no complex conjugation)}$$

Operators, matrices

$$Q = Q_1 \partial_x + Q_0$$

$$Q_1 = \begin{pmatrix} -J & \\ & \end{pmatrix}$$

$$Q_0 = \begin{pmatrix} A & B \\ B & \end{pmatrix}$$

$$A = -\frac{1}{4}(Pp + q_x) Z$$

$$B = \frac{1}{4} e^{\mathrm{i}Rq/2} = \frac{1}{4} \begin{pmatrix} \exp(-q/2) & \\ & \exp(q/2) \end{pmatrix}$$

$$\mathcal{Q} = \begin{pmatrix} T & \\ & T \end{pmatrix} Q \begin{pmatrix} T^{-1} & \\ & T^{-1} \end{pmatrix}$$

$$\mathcal{A} = TAT^{-1}$$

$$\mathcal{B} = TBT^{-1}$$

$$E_\omega = \begin{pmatrix} \cos(\omega(\lambda)x) & \sin(\omega(\lambda)x) \\ -\sin(\omega(\lambda)x) & \cos(\omega(\lambda)x) \end{pmatrix}$$

$$P = \sqrt{1 - \partial_x^2}$$

$$I = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$$

$$J = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$$

$$R = \begin{pmatrix} \mathrm{i} & \\ & -\mathrm{i} \end{pmatrix}$$

$$Z = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$$

$$T = \begin{pmatrix} 1 & \mathrm{i} \\ 1 & -\mathrm{i} \end{pmatrix}$$

$$M = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}$$

$$M_1 = \begin{pmatrix} m_1 \\ m_3 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} m_2 \\ m_4 \end{pmatrix}$$

$$\mathcal{M} = TMT^{-1}$$

$$\mathcal{E}_\omega = \begin{pmatrix} e^{-\mathrm{i}\omega(\lambda)x} & \\ & e^{\mathrm{i}\omega(\lambda)x} \end{pmatrix}$$

Spectral ordering

$$a \preceq b \quad \Leftrightarrow \quad \Big[|a| < |b|\Big] \quad \text{or} \quad \Big[|a| = |b| \quad \text{and} \quad \mathrm{Im}a \le \mathrm{Im}b\Big].$$

References

[1] Belokolos, E., Bobenko, A., Enol'skij, V., Its, A., and Matveev, V. *Algebro-geometric*

approach to nonlinear integrable equations. Springer Series in Nonlinear Dynamics. Springer, Berlin, 1994.

[2] Bour, E. Théorie de la déformation des surfaces. *J. L'Ecole Impériale Polytechnique* (1862).

[3] Djakov, P., and Mityagin, B. S. Instability zones of periodic 1-dimensional Schrödinger and Dirac operators. *Russian Mathematical Surveys 61*, 4 (2006), 663–766.

[4] Faddeev, L., Takhtajan, L., and Zakharov, V. Complete description of solutions of the sine-Gordon equation. *Dokl.Akad.Nauk Ser.Fiz. 219* (1974), 1334–1337.

[5] Faddeev, L. Takhtajan, L. *Hamiltonian Methods in the Theory of Solitons.* Classics in Mathematics. Springer, Berlin, 2007.

[6] Grébert, B., and Kappeler, T. *The Defocusing NLS Equation and Its Normal Form.* Lectures in Mathematics. European Mathematical Society, 2014.

[7] Kappeler, T., and Mityagin, B. Estimates for periodic and Dirichlet eigenvalues of the Schrödinger operator. *SIAM Journal on Mathematical Analysis 33*, 1 (2001), 113–152.

[8] Kappeler, T., and Pöschel, J. *KdV & KAM.* Ergebnisse der Mathematik und ihrer Grenzgebiete. Springer, Berlin, 2003.

[9] Kappeler, T., Serier, F., and Topalov, P. On the characterization of the smoothness of skew-adjoint potentials in periodic Dirac operators. *Journal of Functional Analysis 256*, 7 (2009), 2069 – 2112.

[10] Kappeler, T., and Topalov, P. Global wellposedness of KdV in $H^{-1}(\mathbb{T},\mathbb{R})$. *Duke Math. J. 135*, 2 (2006), 327–360.

[11] Kappeler, T., and Topalov, P. On normalized differentials on hyperelliptic curves of infinite genus. *J. Differential Geom. 105*, 2 (2017), 209–248.

[12] Krichever, I. M. Spectral theory of two-dimensional periodic operators and its applications. *Russian Mathematical Surveys 44*, 2 (1989), 145–225.

[13] Kuksin, S. *Analysis of Hamiltonian PDEs.* Oxford Lecture Series in Mathematics and its Applications. Oxford University Press, 2000.

[14] Lamb, J. G. L. *Elements of soliton theory.* John Wiley & Sons, Inc., New York, 1980.

[15] Lamb Jr, G. Analytical descriptions of ultrashort optical pulse propagation in a resonant medium. *Reviews of Modern Physics 43*, 2 (1971), 99.

[16] Lebwohl, P., and Stephen, M. J. Properties of vortex lines in superconducting barriers. *Phys. Rev. 163* (Nov 1967), 376–379.

[17] McKean, H. The sine-Gordon and sinh-Gordon equations on the circle. *Communications on Pure and Applied Mathematics 34*, 2 (1981), 197–257.

[18] McKean, H., and Vaninsky, K. Action-angle variables for the cubic Schrödinger equation. *Communications on Pure and Applied Mathematics 50*, 6 (1997), 489–562.

[19] Pöschel, J. Hill's potentials in weighted Sobolev spaces and their spectral gaps. *Mathematische Annalen 349*, 2 (2011), 433–458.

[20] Skyrme, T. H. R. Particle states of a quantized meson field. *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences 262*, 1309 (1961), 237–245.

[21] Widmer, Y. On the integrability of the sine-Gordon equation. *ArXiv e-prints* (Aug. 2015).